ERROR ESTIMATES AND CONVERGENCE RATES FOR FILTERED BACK PROJECTION

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Abstract. We consider the approximation of target functions from fractional Sobolev spaces by the method of filtered back projection (FBP), which gives an inversion of the Radon transform. The objective of this paper is to analyse the intrinsic FBP approximation error which is incurred by the use of a low-pass filter with finite bandwidth. To this end, we prove L^2-error estimates on Sobolev spaces of fractional order. The obtained error bounds are affine-linear with respect to the distance between the filter's window function and the constant function 1 in the L^∞-norm. With assuming more regularity of the window function, we refine the error estimates to prove convergence for the FBP approximation in the L^2-norm as the filter's bandwidth goes to infinity. Further, we determine asymptotic convergence rates in terms of the bandwidth of the low-pass filter and the smoothness of the target function. Finally, we develop convergence rates for noisy data, where we first prove estimates for the data error, which we then combine with our estimates for the approximation error.

1. Introduction

The method of *filtered back projection* (FBP) is a popular reconstruction technique for bivariate functions from Radon data. To formulate the basic reconstruction problem mathematically, we regard for $f \in \mathrm{L}^1(\mathbb{R}^2)$ its *Radon transform*

$$\mathcal{R}f(t,\theta) = \int_{\{x\cos(\theta)+y\sin(\theta)=t\}} f(x,y)\,\mathrm{d}(x,y) \qquad \text{for } (t,\theta) \in \mathbb{R}\times[0,\pi).$$

Here, the set $\{(x,y) \mid x\cos(\theta)+y\sin(\theta)=t\} \subset \mathbb{R}^2$ describes the straight line $\ell_{t,\theta}$ with distance t to the origin that is perpendicular to the unit vector $\mathbf{n}_\theta = (\cos(\theta),\sin(\theta))^T$. Note that the Radon transform $\mathcal{R}$ maps a bivariate function $f \equiv f(x,y)$ in Cartesian coordinates onto a bivariate function $\mathcal{R}f \equiv \mathcal{R}f(t,\theta)$ in polar coordinates.

Now the reconstruction problem reads as follows.

Problem 1.1 (Basic reconstruction problem). *On given domain $\Omega \subseteq \mathbb{R}^2$, reconstruct a bivariate function $f \in \mathrm{L}^1(\Omega)$ on Ω from Radon data*

$$\{\mathcal{R}f(t,\theta) \mid t \in \mathbb{R},\, \theta \in [0,\pi)\}.$$

Therefore, the basic reconstruction problem seeks for the inversion of the Radon transform $\mathcal{R}$, which is accomplished by the method of filtered back projection. For a comprehensive mathematical treatment of the Radon transform and its inversion, we refer to the textbooks [4, 17].

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The outline of this paper is as follows. In §2 we discuss the main ingredients of the filtered back projection (FBP) and their relevant analytical properties. In particular, we explain in §2 how suitable low-pass filters can be used to obtain an approximate FBP reconstruction method for bivariate functions. The aim of this paper is to analyse the approximation error of the FBP reconstruction, whose error bounds depend on the low-pass filter's window function, on its bandwidth and on the regularity of the target function. Before doing so, we first describe other related methods in §3 and explain their differences to our approach. Further, in §4 we recall an error estimate from our previous work [1] concerning the FBP reconstruction error in the L^2-norm for the relevant case of target functions from Sobolev spaces of fractional order.

That error estimate from [1] allows us to show convergence of the approximate reconstruction to the target function as the filter's bandwidth goes to infinity, but only under rather strong assumptions. In contrast, due to a result by Madych [11], convergence can be shown under much weaker assumptions. This has motivated us to investigate the refinement of our previous L^2-error estimate, as detailed in §5. On the basis of our refined error estimates we are able to prove convergence under much weaker conditions. Furthermore, this allows us to determine asymptotic rates of convergence in terms of the bandwidth of the low-pass filter and the smoothness of the target function. In §6 and §7 we show that the convergence rate saturates with respect to the differentiability order of the filter's window function. Our theoretical results are supported by numerical simulations.

Finally, in §8 we develop deterministic convergence rates for noisy data, where we combine our estimates on the approximation error from §5 and §7 with error estimates for the data error. In particular, we estimate the norm of the regularization operator in dependence of the regularization parameter. This allows us to balance the approximation error with the noise level and the norm of the regularization operator, as we elaborate on in detail in the final step of §8.

2. Filtered Back Projection

The inversion of the Radon transform $\mathcal{R}$ is well understood and involves the (continuous) *Fourier transform*, here taken as

$$\mathcal{F}g(S,\theta) = \int_{\mathbb{R}} g(t,\theta)\mathrm{e}^{-\mathrm{i}tS}\,\mathrm{d}t \quad \text{for } (S,\theta) \in \mathbb{R}\times[0,\pi)$$

for $g \equiv g(t,\theta)$ in polar coordinates satisfying $g(\cdot,\theta) \in \mathrm{L}^1(\mathbb{R})$ for all $\theta \in [0,\pi)$, as well as the *back projection*

$$\mathcal{B}h(x,y) = \frac{1}{\pi}\int_0^\pi h(x\cos(\theta) + y\sin(\theta),\theta)\,\mathrm{d}\theta \quad \text{for } (x,y) \in \mathbb{R}^2$$

for $h \in \mathrm{L}^1(\mathbb{R}\times[0,\pi))$. Note that the back projection $\mathcal{B}$ maps a bivariate function $h \equiv h(t,\theta)$ in polar coordinates onto a bivariate function $\mathcal{B}h \equiv \mathcal{B}h(x,y)$ in Cartesian coordinates.

Later in this work we also use the (continuous) *Fourier transform* on $\mathbb{R}^2$, defined as

$$\mathcal{F}f(X,Y) = \int_{\mathbb{R}}\int_{\mathbb{R}} f(x,y)\mathrm{e}^{-\mathrm{i}(xX+yY)}\,\mathrm{d}x\,\mathrm{d}y \quad \text{for } (X,Y) \in \mathbb{R}^2$$

for $f \equiv f(x,y)$ in Cartesian coordinates, where $f \in \mathrm{L}^1(\mathbb{R}^2)$.

Now the inversion of the Radon transform is given by the *filtered back projection formula* [3, 17]

$$f(x,y) = \frac{1}{2}\,\mathcal{B}\big(\mathcal{F}^{-1}[|S|\mathcal{F}(\mathcal{R}f)(S,\theta)]\big)(x,y) \quad \forall\,(x,y) \in \mathbb{R}^2, \tag{2.1}$$

which holds for any function $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathcal{C}(\mathbb{R}^2)$.

We remark that the FBP formula is numerically *unstable*. Indeed, by applying the filter $|S|$ to the Fourier transform $\mathcal{F}(\mathcal{R}f)$ in (2.1), especially the high frequency components of $\mathcal{R}f$ are amplified by the magnitude of $|S|$. To stabilize the FBP reconstruction method, we follow a standard approach and replace the filter $|S|$ in (2.1) by a *low-pass filter* A_L of the form

$$A_L(S) = |S| W(S/L)$$

with finite *bandwidth* $L > 0$ and an even *window function* $W : \mathbb{R} \longrightarrow \mathbb{R}$ with compact support $\mathrm{supp}(W) \subseteq [-1, 1]$. Further, we assume $W \in \mathrm{L}^\infty(\mathbb{R})$.

Therefore, the scaled window function $W_L(S) = W(S/L)$ is even and compactly supported with $\mathrm{supp}(W_L) \subseteq [-L, L]$. In particular, $W_L \in \mathrm{L}^1(\mathbb{R})$, and so, unlike $|S|$, any low-pass filter of the form $A_L(S) = |S| W_L(S)$ is in $\mathrm{L}^1(\mathbb{R})$. When replacing the filter $|S|$ in (2.1) by a low-pass filter $A_L(S)$, the reconstruction of f is no longer exact and we obtain an *approximate FBP reconstruction* f_L via

$$f_L(x, y) = \frac{1}{2} \mathcal{B}\big(\mathcal{F}^{-1}[A_L(S) \mathcal{F}(\mathcal{R}f)(S, \theta)]\big)(x, y) \quad \text{for } (x, y) \in \mathbb{R}^2.$$

For target functions $f \in \mathrm{L}^1(\mathbb{R}^2)$ the approximate reconstruction f_L is defined almost everywhere on $\mathbb{R}^2$ (see [2, Proposition 3.1]) and can be simplified to

$$f_L = \frac{1}{2} \mathcal{B}(q_L * \mathcal{R}f), \tag{2.2}$$

where $*$ denotes the convolution product of bivariate functions in polar coordinates, given by

$$(q_L * \mathcal{R}f)(S, \theta) = \int_{\mathbb{R}} q_L(t, \theta)\, \mathcal{R}f(S - t, \theta) \,\mathrm{d}t \quad \text{for } (S, \theta) \in \mathbb{R} \times [0, \pi),$$

and where we define the band-limited function $q_L : \mathbb{R} \times [0, \pi) \longrightarrow \mathbb{R}$ as

$$q_L(S, \theta) = \mathcal{F}^{-1} A_L(S) \quad \text{for } (S, \theta) \in \mathbb{R} \times [0, \pi).$$

Note that the function q_L is well-defined on $\mathbb{R} \times [0, \pi)$ and satisfies $q_L \in \mathrm{L}^2(\mathbb{R} \times [0, \pi))$.

Moreover, the approximate FBP reconstruction f_L belongs to $\mathrm{L}^2(\mathbb{R}^2)$ (see [2, Proposition 4.2]) and can be rewritten in terms of the target function f via

$$f_L = f * K_L, \tag{2.3}$$

where $*$ now denotes the convolution product of bivariate functions in Cartesian coordinates, given by

$$(f * K_L)(x, y) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(X, Y)\, K_L(x - X, y - Y) \,\mathrm{d}X \,\mathrm{d}Y \quad \text{for } (x, y) \in \mathbb{R}^2,$$

and where we define the *convolution kernel* $K_L : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as

$$K_L(x, y) = \frac{1}{2} \mathcal{B} q_L(x, y) \quad \text{for } (x, y) \in \mathbb{R}^2.$$

Note that K_L is well-defined on $\mathbb{R}^2$ and satisfies $K_L \in \mathcal{C}_0(\mathbb{R}^2) \cap \mathrm{L}^2(\mathbb{R}^2)$ (see [2, Proposition 4.1]). Further, its Fourier transform is given by

$$\mathcal{F} K_L(x, y) = W_L(x, y) \quad \text{for almost all } (x, y) \in \mathbb{R}^2, \tag{2.4}$$

where the compactly supported and radially symmetric bivariate window function $W_L \in \mathrm{L}^\infty(\mathbb{R}^2)$ is defined as

$$W_L(x, y) = W_L\big(\sqrt{x^2 + y^2}\big) \quad \text{for } (x, y) \in \mathbb{R}^2.$$

Rigorous derivations of the formulas (2.2), (2.3) and (2.4) can be found, for example, in [2], where no additional assumptions on the band-limited function q_L and the convolution kernel K_L

are needed. We remark that relation (2.3) was derived via ridge functions in [11, Proposition 1] under different assumptions.

For the sake of brevity, we call any application of the approximate FBP formula (2.2) an *FBP method*. Therefore, each FBP method provides one approximation f_L to f, $f_L \approx f$, whose quality depends on the choice of the low-pass filter A_L.

In the following, we analyse the intrinsic error of the FBP method which is incurred by the use of the low-pass filter A_L, i.e., we wish to analyse the reconstruction error

$$e_L = f - f_L \tag{2.5}$$

with respect to the filter's window function W and bandwidth L. To this end, in §4 and §5 we prove L^2-error estimates on e_L for target functions f from Sobolev spaces of fractional order. Here, the *Sobolev space* $\mathrm{H}^\alpha(\mathbb{R}^2)$ of order $\alpha \in \mathbb{R}$, defined as

$$\mathrm{H}^\alpha(\mathbb{R}^2) = \left\{ f \in \mathcal{S}'(\mathbb{R}^2) \mid \|f\|_\alpha < \infty \right\}, \tag{2.6}$$

is equipped with the norm $\|\cdot\|_\alpha$, where

$$\|f\|_\alpha^2 = \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(1 + x^2 + y^2\right)^\alpha \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y,$$

and where $\mathcal{S}'(\mathbb{R}^2)$ in (2.6) denotes the Schwartz space of tempered distributions on $\mathbb{R}^2$.

In relevant applications of (medical) image processing, Sobolev spaces of compactly supported functions,

$$\mathrm{H}_0^\alpha(\Omega) = \left\{ f \in \mathrm{H}^\alpha(\mathbb{R}^2) \mid \mathrm{supp}(f) \subseteq \overline{\Omega} \right\},$$

on an open and bounded domain $\Omega \subset \mathbb{R}^2$, and of fractional order $\alpha > 0$ play an important role (cf. [16]). In fact, the density function f of an image in $\Omega \subset \mathbb{R}^2$ has usually jumps along smooth curves, but is otherwise smooth off these curve singularities. Such functions belong to the Sobolev space $\mathrm{H}_0^\alpha(\mathbb{R}^2)$ for $\alpha < \frac{1}{2}$. Consequently, we can consider the density of an image as a function in a Sobolev space $\mathrm{H}_0^\alpha(\Omega)$ whose order α is close to $\frac{1}{2}$.

3. Summability Methods, Approximate Inverse, and Other Related Approaches

Before we develop our L^2-error estimates and convergence rates, we first discuss related methods, where we explain how their results differ from ours. We adapt the notations to our setting.

3.1. Summability Methods.

In [11], Madych describes the reconstruction of functions from Radon data based on summability formulas. The basic idea is to choose a convolution kernel $K : \mathbb{R}^2 \longrightarrow \mathbb{R}$ as an approximation of the identity and to compute the convolution product $f * K_L$ to approximate the target function f, where, for $L > 0$, the scaled kernel K_L is given by

$$K_L(x,y) = L^2 \, K(Lx, Ly) \quad \text{for } (x,y) \in \mathbb{R}^2.$$

If K is chosen to be a uniform sum of ridge functions, the convolution $f * K_L$ can be expressed in terms of the Radon data $\mathcal{R}f$ as in the approximate FBP formula (2.2), see [11, Proposition 1]. Convolution kernels K that can be represented as uniform sums of ridge functions are characterized in [11, Section 2.2]. Moreover, for target functions $f \in \mathrm{L}^p(\mathbb{R}^2)$, $1 \le p \le \infty$, the reconstruction error

$$f - f * K_L$$

is estimated in terms of the L^p-modulus of continuity $\omega_p(f;\delta)$, where, for $\delta > 0$,

$$\omega_p(f;\delta) = \sup_{\|(X,Y)\|_2 \leq \delta} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-X, y-Y) - f(x,y)|^p \, \mathrm{d}x \, \mathrm{d}y \right)^{1/p} \quad \text{for } 1 \leq p < \infty$$

and

$$\omega_\infty(f;\delta) = \sup_{\|(X,Y)\|_2 \leq \delta} \operatorname*{ess\,sup}_{(x,y)\in\mathbb{R}^2} |f(x-X, y-Y) - f(x,y)|.$$

Under the assumption that K_L, for $L > 0$, is a family of integrable convolution kernels satisfying

$$\int_{\mathbb{R}} \int_{\mathbb{R}} K_L(x,y) \, \mathrm{d}x \, \mathrm{d}y = 1$$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |K_L(x,y)| \, \mathrm{d}x \, \mathrm{d}y \leq c_0$$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \sqrt{x^2+y^2} \, |K_L(x,y)| \, \mathrm{d}x \, \mathrm{d}y \leq c_1 \, L^{-1}$$

for some constants $c_0, c_1 \in \mathbb{R}_{\geq 0}$ independent of L, it is shown in [11, Proposition 7] that

$$\|f - f * K_L\|_{\mathrm{L}^p(\mathbb{R}^2)} \leq c \, \omega_p(f; L^{-1}), \tag{3.1}$$

where the constant $c \in \mathbb{R}_{\geq 0}$ is independent of f and L. The proof is based on direct calculations in the cases $p = 1$ and $p = \infty$, and on the integral variant of Minkowski's inequality for $1 < p < \infty$.

To exploit a higher order moment condition on the convolution kernel K_L,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \sqrt{x^2+y^2}^{\,k} \, |K_L(x,y)| \, \mathrm{d}x \, \mathrm{d}y \leq c_k \, L^{-k}$$

for some integer $k \geq 2$ and a constant $c_k \in \mathbb{R}_{\geq 0}$ independent of L, the modified kernels

$$\tilde{K}_L^k(x,y) = \sum_{j=0}^{k-1} (-1)^{k-j-1} \, \frac{k!}{(k-j)! \, j!} \, (k-j)^{-2} \, K_L(x/(k-j), y/(k-j)) \quad \text{for } (x,y) \in \mathbb{R}^2$$

are defined and the corresponding reconstruction error

$$f - f * \tilde{K}_L^k$$

is estimated in terms of the k-th order L^p-modulus of smoothness $\omega_p^k(f;\delta)$ of f via

$$\|f - f * \tilde{K}_L^k\|_{\mathrm{L}^p(\mathbb{R}^2)} \leq c \, \omega_p^k(f; L^{-1}), \tag{3.2}$$

where the constant $c \in \mathbb{R}_{\geq 0}$ is again independent of f and L, see [11, Proposition 8].

The constant c in the estimates (3.1) and (3.2) depends on the L^1-norm of the convolution kernel K_L, so that the assumption $K_L \in \mathrm{L}^1(\mathbb{R}^2)$ is essential and cannot be omitted. However, the integrability of K_L implies that its Fourier transform $\mathcal{F}K_L$ is continuous on $\mathbb{R}^2$.

In our setting the condition $K_L \in \mathrm{L}^1(\mathbb{R}^2)$ would imply that the univariate window function $W \in \mathrm{L}^\infty(\mathbb{R})$ is continuous on $\mathbb{R}$, due to formula (2.4). But unlike in [11], we merely require that W has compact support with $\mathrm{supp}(W) \subseteq [-1,1]$, where we essentially want to allow discontinuities for W at the boundary points of $[-1,1]$. Therefore, the assumptions on K_L in [11] lead, in comparison with this paper, to more restrictive conditions on W.

In [12], Madych considers two particular choices for the convolution kernel K_L, where the first one yields a natural approximation of Radon's classical reconstruction formula from [18], whereas the second one leads to an approximation of an alternative inversion formula derived in [11, Corollary 2].

For these two choices of kernels K_L and for target functions $f \in \mathrm{L}^\infty(\mathbb{R}^2)$ that are Hölder continuous of order $\alpha > 0$ at $(x,y) \in \mathbb{R}^2$, the pointwise reconstruction error

$$f(x,y) - (f * K_L)(x,y)$$

is in [12] estimated in terms of the parameter L of the scaled kernel K_L and the Hölder exponent α of the target function f. Again, the so obtained estimates in [12] rely on the assumption $K_L \in \mathrm{L}^1(\mathbb{R}^2)$, and so they do not apply to the setting of this paper.

3.2. **Approximate Inverse.** The method of approximate inverse was developed by Louis and Maass in [9] to solve ill-posed linear operator equations of the form

$$Af = g$$

for $f \in \mathcal{X}$, where $A : \mathcal{X} \longrightarrow \mathcal{Y}$ is a continuous linear operator between Hilbert spaces $\mathcal{X}$ and $\mathcal{Y}$, and where $g \in \mathcal{Y}$ are viewed as input measurements. In the setting of this paper, the operator A is the Radon transform $\mathcal{R}$.

Now the basic idea of the approximate inverse [9] is to select a smoothing operator $E_\gamma : \mathcal{X} \longrightarrow \mathcal{X}$, for $\gamma > 0$, to compute a smoothed version

$$f_\gamma = E_\gamma f$$

of the target function f. If $\mathcal{X}$ is a space of real-valued functions on a domain Ω, this is done by calculating the moments

$$f_\gamma(x) = (f, e_\gamma(x,\cdot))_{\mathcal{X}} \quad \text{for } x \in \Omega$$

with a suitable family of mollifiers $e_\gamma : \Omega \times \Omega \longrightarrow \mathbb{R}$ satisfying

$$\lim_{\gamma \to 0} \|f - f_\gamma\|_{\mathcal{X}} = 0.$$

The computation of f_γ from the given data $g \in \mathcal{Y}$ is achieved by approximating $e_\gamma(x,\cdot)$ in the range of the adjoint operator A^* by the *reconstruction kernel* $v_\gamma(x) \in \mathcal{Y}$ solving

$$\min_{v \in \mathcal{Y}} \|A^* v - e_\gamma(x,\cdot)\|_{\mathcal{X}}$$

so that

$$f_\gamma(x) = (f, e_\gamma(x,\cdot))_{\mathcal{X}} \approx (g, v_\gamma(x))_{\mathcal{Y}} \quad \text{for } x \in \Omega.$$

The mapping $S_\gamma : \mathcal{Y} \longrightarrow \mathcal{X}$, defined as

$$S_\gamma g(x) = (g, v_\gamma(x))_{\mathcal{Y}} \quad \text{for } x \in \Omega,$$

is then called the *approximate inverse* of the operator A.

Now the application of the approximate inverse to the Radon transform, i.e., $A = \mathcal{R}$, yields a reconstruction formula of the filtered back projection type. For detailed investigations on properties of the approximate inverse and its relation to other regularization methods we refer to Louis [7, 8].

Jonas and Louis [6] consider the case where $\mathcal{X}$ and $\mathcal{Y}$ are L^2-spaces and where A is an operator with smoothing index $\alpha > 0$ in the Sobolev scale, i.e., there exist constants $c_1, c_2 > 0$ satisfying

$$c_1 \|f\|_{\mathrm{L}^2} \leq \|Af\|_{\mathrm{H}^\alpha} \leq c_2 \|f\|_{\mathrm{L}^2} \quad \forall\, f \in \mathcal{N}(A)^\perp.$$

For mollifiers e_γ of convolution type,

$$e_\gamma(x,y) = e_\gamma(x-y) \quad \forall\, x,y \in \Omega,$$

sufficient conditions are then derived under which the approximate inverse S_γ yields a regularization method of optimal order, cf. [6, Theorems 3.2 & 5.3]. The proofs in [6] rely on similar techniques that we use in §5 to obtain our refined error estimates. Moreover, in the proof of [6, Theorem 5.3], the estimate

$$\|f - f_\gamma\|_{\mathrm{L}^2} \leq c_\beta \, \gamma^{\theta \frac{\beta}{\alpha}} \, \|f\|_{\mathrm{H}^\beta} \quad \forall \, f \in \mathrm{H}^\beta \tag{3.3}$$

is shown for all $0 < \beta \leq \beta^*$, for some constant $c_\beta > 0$. To this end, it is assumed that there are positive constants $\theta, \beta^*, c_{\beta^*} > 0$ satisfying

$$\sup_\xi \left\{ \left(1 + \|\xi\|_2^2\right)^{-\beta^*/2} \left| (2\pi)^{n/2} \, \mathcal{F} e_\gamma(\xi) - 1 \right| \right\} \leq c_{\beta^*} \, \gamma^{\theta \frac{\beta^*}{\alpha}} \quad \forall \, \gamma > 0. \tag{3.4}$$

But (3.4) is in [6] verified merely for one prototypical case, where the mollifier e_γ is a sinc function.

In contrast to this, we develop concrete and easy-to-check conditions on the window function W which guarantee that the inherent FBP reconstruction error e_L in (2.5) behaves in the fashion of estimate (3.3). Moreover, our estimates in §6 and §7 allow for nontrivial statements concerning the behaviour of the reconstruction error in the case $\beta > \beta^*$.

Louis and Schuster [10] apply the method of approximate inverse to the Radon transform $\mathcal{R}$ to derive inversion formulas for the parallel beam geometry in computerized tomography. They consider both the continuous and discrete setting, where they explain how to compute the reconstruction kernel for a chosen mollifier. This yields inversion formulas of filtered back projection type. The approach in [10] relies, for finite data sets, on a truncation of the singular value decomposition of $\mathcal{R}$. But [10] contains no results concerning error estimates or convergence rates, unlike this paper.

Rieder and Schuster [23, 24] focus on semi-discrete systems

$$A_n f = g_n,$$

where the semi-discrete operator $A_n : \mathcal{X} \longrightarrow \mathbb{C}^n$ and the measurements $g_n \in \mathbb{C}^n$ are defined via an observation operator $\Psi_n : \mathcal{Y} \longrightarrow \mathbb{C}^n$ by

$$A_n = \Psi_n A \quad \text{and} \quad g_n = \Psi_n g.$$

Their work in [23, 24] proposes a technique for approximating the discrete reconstruction kernel for a given mollifier. Moreover, they prove convergence for the resulting discrete version of the approximate inverse. Finally, they apply their results to the Radon transform to obtain convergence rates for the discrete filtered back projection algorithm in parallel beam geometry, as the discretization parameters go to zero. Concrete examples of mollifier/reconstruction kernel pairs for the Radon transform are given in [20].

Since the approach in [23, 24] considers the semi-discrete setting, the method parameter $\gamma > 0$ is necessarily coupled with the discretization parameters. Therefore the intrinsic approximation error

$$f - f_\gamma$$

for the continuous approximate inverse reconstruction f_γ of f (i.e., for complete Radon data) is not considered explicitly, unlike in this paper.

In particular, the results of this paper are not covered by the theory of Rieder and Schuster. Moreover, in [23, 24] the mollifier is required to have compact support. In contrast, we assume the window function W to be compactly supported with $\mathrm{supp}(W) \subseteq [-1, 1]$. Due to formula (2.4) and the Paley-Wiener theorem this implies that the convolution kernel K_L cannot have compact support. Therefore, the setting of this paper is essentially different from that in [23, 24].

The results in [24] lead to suboptimal convergence rates for the discrete filtered back projection algorithm, as this is explained in [24] (cf. the paragraph after [24, Corollary 5.6]). But Rieder and

Faridani [21] prove optimal L^2-convergence rates for a semi-discrete filtered back projection algorithm in parallel beam geometry, where no discretization of the back projection operator $\mathcal{B}$ is considered. This is incorporated by Rieder and Schneck in [22] leading to optimal L^2-convergence rates for a fully discrete version of the filtered back projection algorithm, and for sufficiently smooth f. We remark that the resulting representation of the discretized approximate FBP formula depends on the utilized filter function, the interpolation method and the discretization parameters. Therefore, the inherent FBP reconstruction error

$$e_L = f - f_L,$$

incurred by a low-pass filter of finite bandwidth L, is not estimated in [21, 22], unlike in this paper.

3.3. **Other related Approaches.** Raviart [19] analyses the reconstruction error

$$f - f * K_L$$

for target functions f from Sobolev spaces of integer order, i.e., $f \in \mathrm{H}^{m,p}(\mathbb{R}^2)$ for some $m \in \mathbb{N}_0$ and $1 \leq p \leq \infty$, where

$$\mathrm{H}^{m,p}(\mathbb{R}^2) = \big\{ f \in \mathrm{L}^p(\mathbb{R}^2) \mid \|f\|_{m,p} < \infty \big\}$$

with

$$\|f\|_{m,p} = \begin{cases} \left(\sum_{\alpha+\beta \leq m} \left\| \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} f \right\|^p_{\mathrm{L}^p(\mathbb{R}^2)} \right)^{1/p} & \text{for } 1 \leq p < \infty \\ \max_{\alpha+\beta \leq m} \left\| \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} f \right\|_{\mathrm{L}^\infty(\mathbb{R}^2)} & \text{for } p = \infty. \end{cases}$$

In the approach taken in [19], the convolution kernel K is required to satisfy $K \in \mathcal{C}(\mathbb{R}^2) \cap \mathrm{L}^1(\mathbb{R}^2)$ and, for some integer $k \in \mathbb{N}$,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} K(x,y) \, \mathrm{d}x \, \mathrm{d}y = 1$$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} x^\alpha \, y^\beta \, K(x,y) \, \mathrm{d}x \, \mathrm{d}y = 0 \quad \forall \, \alpha, \beta \in \mathbb{N}_0 : \; 1 \leq \alpha + \beta \leq k-1 \tag{3.5}$$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \sqrt{x^2+y^2}^{\,k} \, |K(x,y)| \, \mathrm{d}x \, \mathrm{d}y < \infty. \tag{3.6}$$

For $f \in \mathrm{H}^{k,p}(\mathbb{R}^2)$, $1 \leq p \leq \infty$, [19, Lemma I.4.4] then yields error estimates of the form

$$\|f - f * K_L\|_{\mathrm{L}^p(\mathbb{R}^2)} \leq C \, L^{-k} \, |f|_{k,p} \tag{3.7}$$

for some constant $C > 0$ and where

$$|f|_{k,p} = \begin{cases} \left(\sum_{\alpha+\beta = k} \left\| \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} f \right\|^p_{\mathrm{L}^p(\mathbb{R}^2)} \right)^{1/p} & \text{for } 1 \leq p < \infty \\ \max_{\alpha+\beta = k} \left\| \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} f \right\|_{\mathrm{L}^\infty(\mathbb{R}^2)} & \text{for } p = \infty. \end{cases}$$

We remark that the proof for (3.7) in [19] relies on a Taylor expansion of f. Further, the required differentiability order of f is coupled with the k-th order moment conditions on K in (3.5) and (3.6). But the moment condition in (3.6), in combination with the integrability of the kernel K, implies that the Fourier transform $\mathcal{F}K$ is k-times continuously differentiable on $\mathbb{R}^2$. In the setting of this paper, this would require the univariate window function $W \in \mathrm{L}^\infty(\mathbb{R})$ to be k-times continuously differentiable on $\mathbb{R}$, due to formula (2.4).

Therefore, the assumptions on K in [19] lead, in comparison with the approach of this paper, to rather restrictive conditions on W. Indeed, in §6 and §7 we prove error estimates on e_L, where we only assume that the compactly supported window $W \in \mathrm{L}^\infty(\mathbb{R})$, with $\mathrm{supp}(W) \subseteq [-1,1]$, is k-times continuously differentiable on $[-1,1]$, but otherwise allow discontinuities for W at the boundary points of $[-1,1]$. Moreover, in our approach, the target function f lies in a fractional Sobolev space $\mathrm{H}^\alpha(\mathbb{R}^2)$, for $\alpha > 0$, where the smoothness α of f is not coupled with the differentiability order k of W. The constants appearing in our error estimates on $e_L = f - f_L$ are given explicitly, unlike for the error estimate (3.7) from [19].

Finally, we prove saturation for the decay rate of the error bound at order k. In the saturation case, the constant $c_{\alpha,k}$, as explicitly given in our error bound, is strictly monotonically decreasing in $\alpha > k$. Therefore, a smoother target function $f \in \mathrm{H}^\alpha(\mathbb{R}^2)$, allows for a better approximation, even in the case of saturation. This behaviour is not covered by the estimates proven in [19].

We finally discuss the related approach of Schomburg [25], who analyses the convergence rates of certain delta sequences in Sobolev spaces of negative fractional order. For a tempered distribution $\phi \in \mathrm{H}^{-\alpha}(\mathbb{R}^2)$ with $\alpha > 1$, which satisfies further assumptions specified in [25, Theorem 1], asymptotic estimates for the error

$$\phi_n - \delta$$

are derived in the $\mathrm{H}^{-\alpha}$-norm. Here, $\delta \in \mathrm{H}^{-\alpha}(\mathbb{R}^2)$ denotes the Dirac delta distribution, given by

$$\langle \delta, \psi \rangle = \psi(0) \quad \text{for } \psi \in \mathcal{S}(\mathbb{R}^2)$$

with the duality pairing $\langle \cdot, \cdot \rangle$ on $\mathrm{H}^{-\alpha}(\mathbb{R}^2) \times \mathrm{H}^\alpha(\mathbb{R}^2)$, and the sequence $(\phi_n)_{n\in\mathbb{N}} \subset \mathrm{H}^{-\alpha}(\mathbb{R}^2)$ is defined via

$$\langle \phi_n, \psi \rangle = \langle \phi, \psi(\cdot/n, \cdot/n) \rangle \quad \text{for } \psi \in \mathcal{S}(\mathbb{R}^2).$$

For an even convolution kernel $K \in \mathrm{L}^2(\mathbb{R}^2)$ the scaled kernels K_L, for $L > 0$, given by

$$K_L(x,y) = L^2\, K(Lx, Ly) \quad \text{for } (x,y) \in \mathbb{R}^2,$$

can be considered as tempered distributions in $\mathrm{H}^{-\alpha}(\mathbb{R}^2)$ with

$$\langle K_L, f \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} K_L(X,Y)\, f(X,Y)\, \mathrm{d}x\, \mathrm{d}y = \int_{\mathbb{R}} \int_{\mathbb{R}} K(X,Y)\, f(X/L, Y/L)\, \mathrm{d}x\, \mathrm{d}y = \langle K, f(\cdot/L, \cdot/L) \rangle.$$

Observing this, the results from [25] can be used to prove asymptotic pointwise error estimates on

$$f - f * K_L$$

for functions $f \in \mathrm{H}^\alpha(\mathbb{R}^2)$, with $\alpha > 1$, in the H^α-norm of f. Indeed, for fixed $(x,y) \in \mathbb{R}^2$, we have

$$\langle K_L(x - \cdot, y - \cdot), f \rangle = \int_{\mathbb{R}} \int_{\mathbb{R}} K_L(x - X, y - Y)\, f(X,Y)\, \mathrm{d}x\, \mathrm{d}y = (f * K_L)(x,y)$$

and

$$\|K_L(x - \cdot, y - \cdot) - \delta(x - \cdot, y - \cdot)\|_{-\alpha} = \|K_L - \delta\|_{-\alpha},$$

so that

$$|f(x,y) - (f * K_L)(x,y)| \leq \|K_L - \delta\|_{-\alpha}\, \|f\|_\alpha \quad \forall\, (x,y) \in \mathbb{R}^2.$$

The constants from the asymptotic error estimates of [25] are generic and not given explicitly. Further, we are interested in error estimates on $f - f * K_L$ for target functions $f \in \mathrm{H}^\alpha(\mathbb{R}^2)$, where the smoothness α of f is only assumed to be positive. Especially the case $0 < \alpha \leq 1$ is of particular interest, as explained at the end of §2, so that the assumption $\alpha > 1$ is too restrictive for our setting.

We finally remark that pointwise and L^∞-error bounds on $e_L = f - f_L$, along with asymptotic pointwise error formulas, are discussed by Munshi in [13] and by Munshi et al. in [14]. Their theoretical results are supported by numerical experiments in [15]. But they assume certain moment conditions on the convolution kernel K, along with rather restrictive differentiability conditions on the target function f, that we can avoid in this paper.

4. Error Analysis

In this section we prove an L^2-error estimate for $e_L = f - f_L$, where the upper bound on the L^2-norm of e_L is split into two error terms, a first term depending on the filter's window function W and a second one depending on its bandwidth $L > 0$. Although the results of this section are already published in [1], it will be quite instructive for the following analysis in this paper to recall the details of our previous error estimates in [1]. We remark that in the present form of Theorem 4.1 we can omit the assumption $K_L \in \mathrm{L}^1(\mathbb{R}^2)$, which implies that the window function W has to be continuous on $\mathbb{R}$.

Theorem 4.1 (L^2-error estimate, see [1, Theorem 1])**.** *Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for some $\alpha > 0$, and let $W \in \mathrm{L}^\infty(\mathbb{R})$ be even with $\mathrm{supp}(W) \subseteq [-1,1]$. Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded above by*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \|1 - W\|_{\infty,[-1,1]} \, \|f\|_{\mathrm{L}^2(\mathbb{R}^2)} + L^{-\alpha} \, \|f\|_\alpha. \tag{4.1}$$

Since we will use some parts of the proof for a refined error analysis, we recall the proof of the theorem for the reader's convenience.

Proof. For $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{L}^2(\mathbb{R}^2)$, we get, by using the Rayleigh–Plancherel theorem,

$$\begin{aligned}\|e_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)} = \|f - f * K_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)} &= \frac{1}{4\pi^2} \, \|\mathcal{F}f - \mathcal{F}f \cdot \mathcal{F}K_L\|^2_{\mathrm{L}^2(\mathbb{R}^2;\mathbb{C})} \\ &= \frac{1}{4\pi^2} \, \|\mathcal{F}f - W_L \cdot \mathcal{F}f\|^2_{\mathrm{L}^2(\mathbb{R}^2;\mathbb{C})},\end{aligned}$$

since, by letting $W_L(x,y) := W_L(r(x,y))$ for $r(x,y) = \sqrt{x^2+y^2}$ and $(x,y) \in \mathbb{R}^2$, we have the identity

$$\mathcal{F}K_L(x,y) = W_L(x,y) \quad \text{for almost all } (x,y) \in \mathbb{R}^2$$

in consequence of [2, Proposition 4.1].

We split the above representation of the L^2-error into a sum of two integrals,

$$\|e_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)} = I_1 + I_2, \tag{4.2}$$

where we let

$$I_1 := \frac{1}{4\pi^2} \int_{r(x,y) \leq L} |(\mathcal{F}f - W_L \cdot \mathcal{F}f)(x,y)|^2 \, \mathrm{d}(x,y), \tag{4.3}$$

$$I_2 := \frac{1}{4\pi^2} \int_{r(x,y) > L} |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y). \tag{4.4}$$

For $W \in \mathrm{L}^\infty(\mathbb{R})$, integral I_1 can be bounded above by

$$I_1 \leq \frac{1}{4\pi^2} \|1 - W_L\|^2_{\infty,[-L,L]} \, \|\mathcal{F}f\|^2_{\mathrm{L}^2(\mathbb{R}^2;\mathbb{C})} = \|1 - W\|^2_{\infty,[-1,1]} \, \|f\|^2_{\mathrm{L}^2(\mathbb{R}^2)}$$

and, for $f \in \mathrm{H}^\alpha(\mathbb{R}^2)$, with $\alpha > 0$, integral I_2 can be bounded above by

$$I_2 \leq \frac{1}{4\pi^2} \int_{r(x,y) > L} \left(1 + x^2 + y^2\right)^\alpha \, L^{-2\alpha} \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y) \leq L^{-2\alpha} \, \|f\|^2_\alpha,$$

which completes the proof. ∎

The above theorem shows that the choices of both the window function W and the bandwidth L are of fundamental importance for the L^2-error of the FBP method. In fact, for fixed target function f and bandwidth L, the obtained error estimate is affine-linear with respect to the distance between the window function W and the constant function 1 in the L^∞-norm on the interval $[-1,1]$. This behaviour has also been observed numerically in [1].

Moreover, the error term $\|1-W\|_{\infty,[-1,1]}$ can be used to evaluate the quality of the window function W. Note that the window $W \equiv \chi_{[-1,1]}$ of the Ram–Lak filter is the unique minimizer of that quality indicator, so that the Ram–Lak filter is in this sense the *optimal* low-pass filter.

Finally, the smoothness of the target function f determines the decay rate of the second error term by

$$L^{-\alpha}\|f\|_\alpha = \mathcal{O}(L^{-\alpha}) \quad \text{for} \quad L \longrightarrow \infty.$$

However, the right hand side of our L^2-error estimate can only tend to zero if we choose the Ram–Lak filter, $W \equiv \chi_{[-1,1]}$, and let the bandwidth L go to ∞.

Nevertheless, the following theorem of Madych [11] shows that we get convergence of the FBP reconstruction f_L in the L^p-norm under weaker assumptions, for target functions $f \in \mathrm{L}^p(\mathbb{R}^2)$ with $1 \leq p < \infty$.

Theorem 4.2 (Convergence in the L^p-norm, see [11, Proposition 5])**.** *Let the convolution kernel $K \equiv K_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfy $K \in \mathrm{L}^1(\mathbb{R}^2)$ with*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} K(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1.$$

Then, for $f \in \mathrm{L}^p(\mathbb{R}^2)$, $1 \leq p < \infty$,

$$\|e_L\|_{\mathrm{L}^p(\mathbb{R}^2)} \longrightarrow 0 \quad \textit{for} \quad L \longrightarrow \infty.$$

For the reader's convenience, we give a proof of the theorem, which relies on Lebegue's theorem on dominated convergence.

Proof. For $f \in \mathrm{L}^p(\mathbb{R}^2)$, $1 \leq p < \infty$, and $(X,Y) \in \mathbb{R}^2$, we define

$$\Delta_f(X,Y) = \|f(\cdot - X, \cdot - Y) - f\|_{\mathrm{L}^p(\mathbb{R}^2)}.$$

Then, we have

$$\Delta_f(X,Y) \longrightarrow 0 \quad \text{for} \quad (X,Y) \longrightarrow (0,0),$$

since this holds for continuous functions f with compact support, i.e., $f \in \mathcal{C}_c(\mathbb{R}^2)$, and $\mathcal{C}_c(\mathbb{R}^2)$ is dense in $\mathrm{L}^p(\mathbb{R}^2)$ for $1 \leq p < \infty$.

Relying on the scaling property

$$K_L(x,y) = L^2\,K(Lx,Ly) \quad \forall\,(x,y) \in \mathbb{R}^2 \tag{4.5}$$

we get

$$\int_{\mathbb{R}}\int_{\mathbb{R}} K_L(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}}\int_{\mathbb{R}} K(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1,$$

and can rewrite the pointwise error

$$e_L(x,y) = (f - f_L)(x,y) \quad \text{for } (x,y) \in \mathbb{R}^2$$

as

$$e_L(x,y) = (f - f * K_L)(x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}} [f(x,y) - f(x-X,y-Y)]\,K_L(X,Y)\,\mathrm{d}X\,\mathrm{d}Y.$$

Using Minkowski's integral inequality we can estimate the L^p-norm of e_L by

$$\begin{aligned}
\|e_L\|_{\mathrm{L}^p(\mathbb{R}^2)} &= \left(\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\int_{\mathbb{R}}[f(x-X,y-Y)-f(x,y)]\,K_L(X,Y)\,\mathrm{d}X\,\mathrm{d}Y\right|^p\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}\\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|f(x-X,y-Y)-f(x,y)|^p\,|K_L(X,Y)|^p\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}\mathrm{d}X\,\mathrm{d}Y\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}|f(x-X,y-Y)-f(x,y)|^p\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}|K_L(X,Y)|\,\mathrm{d}X\,\mathrm{d}Y\\
&= \int_{\mathbb{R}}\int_{\mathbb{R}}\Delta_f(X,Y)\,|K_L(X,Y)|\,\mathrm{d}X\,\mathrm{d}Y.
\end{aligned}$$

Again, by using the scaling property (4.5), we get

$$\|e_L\|_{\mathrm{L}^p(\mathbb{R}^2)} \leq \int_{\mathbb{R}}\int_{\mathbb{R}}\Delta_f(X/L,Y/L)\;|K(X,Y)|\,\mathrm{d}X\,\mathrm{d}Y.$$

Since

$$|\Delta_f(X/L,Y/L)|\;|K(X,Y)| \leq 2\,\|f\|_{\mathrm{L}^p(\mathbb{R}^2)}\,|K(X,Y)|$$

and, by assumption,

$$\int_{\mathbb{R}}\int_{\mathbb{R}}|K(X,Y)|\,\mathrm{d}X\,\mathrm{d}Y < \infty,$$

in combination with

$$\Delta_f(X/L,Y/L) \longrightarrow 0 \quad \text{for} \quad L\longrightarrow\infty,$$

we finally obtain

$$\|e_L\|_{\mathrm{L}^p(\mathbb{R}^2)} \leq \int_{\mathbb{R}}\int_{\mathbb{R}}\Delta_f(X/L,Y/L)\;|K(X,Y)|\,\mathrm{d}X\,\mathrm{d}Y \longrightarrow 0 \quad \text{for} \quad L\longrightarrow\infty$$

by Lebesgue's theorem on dominated convergence. □

5. Refined Error Analysis

According to Theorem 4.2, the L^2-norm of the FBP reconstruction error $f - f_L$ tends to zero as L goes to ∞. On the grounds of our error estimate in (4.1), however, convergence follows only for the Ram–Lak filter, where $W \equiv \chi_{[-1,1]}$. To obtain convergence under weaker conditions, we need to refine our error estimate.

As in Theorem 4.1 we assume $f \in \mathrm{L}^1(\mathbb{R}^2)\cap\mathrm{H}^\alpha(\mathbb{R}^2)$, for $\alpha > 0$, and consider even window functions $W \in \mathrm{L}^\infty(\mathbb{R})$ with compact support $\mathrm{supp}(W) \subseteq [-1,1]$.

For the sake of brevity, we set $r(x,y) = \sqrt{x^2+y^2}$ for $(x,y)\in\mathbb{R}^2$. Recall the representation of the FBP reconstruction error $e_L = f - f_L$ with respect to the L^2-norm in (4.2), by the sum of two integrals, I_1 in (4.3) and I_2 in (4.4), where integral I_2 can be bounded above by

$$I_2 \leq L^{-2\alpha}\,\|f\|_\alpha^2. \tag{5.1}$$

In Theorem 4.1 we derived an upper bound for integral I_1 in terms of the L^2-norm of the target function f. To obtain convergence for a larger class of window functions, we bound I_1 from above, now also with respect to the H^α-norm of f. Indeed, for $f \in \mathrm{H}^\alpha(\mathbb{R}^2)$, with $\alpha > 0$, we can estimate integral I_1 in (4.3) by

$$\begin{aligned} I_1 &= \frac{1}{4\pi^2} \int_{r(x,y) \leq L} |1 - W_L(x,y)|^2 \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y) \\ &= \frac{1}{4\pi^2} \int_{r(x,y) \leq L} \frac{|1 - W_L(x,y)|^2}{(1 + x^2 + y^2)^\alpha} \, \left(1 + x^2 + y^2\right)^\alpha \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y) \\ &\leq \left(\sup_{S \in [-L,L]} \frac{(1 - W_L(S))^2}{(1 + S^2)^\alpha} \right) \frac{1}{4\pi^2} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(1 + x^2 + y^2\right)^\alpha \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y. \end{aligned}$$

Now note that

$$\sup_{S \in [-L,L]} \frac{(1 - W_L(S))^2}{(1 + S^2)^\alpha} = \sup_{S \in [-L,L]} \frac{(1 - W(S/L))^2}{(1 + S^2)^\alpha} = \sup_{S \in [-1,1]} \frac{(1 - W(S))^2}{(1 + L^2 S^2)^\alpha}.$$

Therefore, with letting

$$\Phi_{\alpha,W}(L) = \sup_{S \in [-1,1]} \frac{(1 - W(S))^2}{(1 + L^2 S^2)^\alpha} \qquad \text{for } L > 0$$

we can express the above bound on I_1 as

$$I_1 \leq \left(\sup_{S \in [-1,1]} \frac{(1 - W(S))^2}{(1 + L^2 S^2)^\alpha} \right) \|f\|_\alpha^2 = \Phi_{\alpha,W}(L) \, \|f\|_\alpha^2.$$

Combining our bounds for integrals I_1 and I_2, this finally leads us to the L^2-error estimate

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)}^2 \leq \left(\sup_{S \in [-1,1]} \frac{(1 - W(S))^2}{(1 + L^2 S^2)^\alpha} + L^{-2\alpha} \right) \|f\|_\alpha^2 = \left(\Phi_{\alpha,W}(L) + L^{-2\alpha}\right) \|f\|_\alpha^2.$$

In summary, we have just established the following result.

Theorem 5.1 (Refined L^2-error estimate)**.** *Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for $\alpha > 0$, and let $W \in \mathrm{L}^\infty(\mathbb{R})$ be even with* $\mathrm{supp}(W) \subseteq [-1,1]$*. Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded above by*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(\Phi_{\alpha,W}^{1/2}(L) + L^{-\alpha}\right) \|f\|_\alpha. \tag{5.2}$$

Our next result shows that, under suitable assumptions on the window W, the function $\Phi_{\alpha,W}(L)$ tends to zero as L goes to ∞.

Theorem 5.2 (Convergence of $\Phi_{\alpha,W}$)**.** *Let the window W be continuous on* $[-1,1]$ *and* $W(0) = 1$*. Then, for any $\alpha > 0$,*

$$\Phi_{\alpha,W}(L) = \max_{S \in [-1,1]} \frac{(1 - W(S))^2}{(1 + L^2 S^2)^\alpha} \longrightarrow 0 \quad \textit{for} \quad L \longrightarrow \infty.$$

Note that we require continuity of the compactly supported window function W only on the interval $[-1,1]$. But we allow discontinuities of W at the boundary points of $[-1,1]$.

Proof. For the sake of brevity, we define the function $\Phi_{\alpha,W,L} : [-1,1] \longrightarrow \mathbb{R}$ via

$$\Phi_{\alpha,W,L}(S) = \frac{(1-W(S))^2}{(1+L^2S^2)^\alpha} \quad \text{for } S \in [-1,1].$$

Because W is continuous on $[-1,1]$ and even, $\Phi_{\alpha,W,L}$ attains a maximum on $[-1,1]$, and we have

$$\Phi_{\alpha,W}(L) = \sup_{S\in[-1,1]} \Phi_{\alpha,W,L}(S) = \max_{S\in[-1,1]} \Phi_{\alpha,W,L}(S) = \max_{S\in[0,1]} \Phi_{\alpha,W,L}(S).$$

In the following, let $S^*_{\alpha,W,L} \in [0,1]$ be the smallest maximizer of the even function $\Phi_{\alpha,W,L}$ on $[0,1]$.

Case 1: $S^*_{\alpha,W,L}$ is uniformly bounded away from 0, i.e.,

$$\exists\, c \equiv c(\alpha,W) > 0 \,\forall\, L > 0 : \; S^*_{\alpha,W,L} \geq c,$$

in which case we get

$$0 \leq \Phi_{\alpha,W,L}\big(S^*_{\alpha,W,L}\big) = \frac{\big(1-W(S^*_{\alpha,W,L})\big)^2}{\big(1+L^2(S^*_{\alpha,W,L})^2\big)^\alpha} \leq \frac{\|1-W\|^2_{\infty,[-1,1]}}{(1+L^2c^2)^\alpha} \xrightarrow{L\to\infty} 0.$$

Case 2: $S^*_{\alpha,W,L}$ tends to 0 as L goes to ∞, i.e.,

$$S^*_{\alpha,W,L} \longrightarrow 0 \quad \text{for} \quad L \longrightarrow \infty.$$

Because W is continuous on $[-1,1]$ and satisfies $W(0) = 1$, we have

$$W(S^*_{\alpha,W,L}) \longrightarrow W(0) = 1 \quad \text{for} \quad L \longrightarrow \infty$$

and, consequently,

$$0 \leq \Phi_{\alpha,W,L}\big(S^*_{\alpha,W,L}\big) = \frac{\big(1-W(S^*_{\alpha,W,L})\big)^2}{\big(1+L^2(S^*_{\alpha,W,L})^2\big)^\alpha} \leq \big(1-W(S^*_{\alpha,W,L})\big)^2 \xrightarrow{L\to\infty} 0.$$

Hence, in both cases we have

$$\Phi_{\alpha,W}(L) = \Phi_{\alpha,W,L}\big(S^*_{\alpha,W,L}\big) \longrightarrow 0 \quad \text{for} \quad L \longrightarrow \infty,$$

which completes our proof. □

By combining Theorems 5.1 and 5.2, we can now conclude convergence of the FBP reconstruction f_L in the L^2-norm for a larger class of window functions W.

Corollary 5.3. *Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for some $\alpha > 0$, and $W \in \mathcal{C}([-1,1])$ with $W(0) = 1$. Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ satisfies*

$$\|e_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)} \leq \big(\Phi_{\alpha,W}(L) + L^{-2\alpha}\big)\, \|f\|^2_\alpha \longrightarrow 0 \quad \textit{for} \quad L \longrightarrow \infty.$$

In particular,

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} = o(1) \quad \textit{for} \quad L \longrightarrow \infty.$$

We are now interested in the rate of convergence for the FBP reconstruction error $\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)}$ as L goes to ∞. Thus, we need to determine the decay rate of $\Phi_{\alpha,W}(L)$. To this end, let $S^*_{\alpha,W,L} \in [0,1]$ again denote the smallest maximizer in $[0,1]$ of the even function

$$\Phi_{\alpha,W,L}(S) = \frac{(1-W(S))^2}{(1+L^2S^2)^\alpha} \quad \text{for } S \in [-1,1].$$

In the following analysis, we rely on the following assumption.

Assumption 5.4. *$S^*_{\alpha,W,L}$ is uniformly bounded away from 0, i.e., there exists a constant $c_{\alpha,W} > 0$, such that*

$$S^*_{\alpha,W,L} \geq c_{\alpha,W} \quad \forall\, L > 0.$$

Under this assumption, we can conclude

$$\Phi_{\alpha,W}(L) = \Phi_{\alpha,W,L}\big(S^*_{\alpha,W,L}\big) \leq \frac{\|1-W\|^2_{\infty,[-1,1]}}{\big(1 + L^2\, c^2_{\alpha,W}\big)^{\alpha}} \leq c_{\alpha,W}^{-2\alpha}\, \|1-W\|^2_{\infty,[-1,1]}\, L^{-2\alpha},$$

in which case we obtain

$$\|e_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)} \leq \Big(c_{\alpha,W}^{-2\alpha}\, \|1-W\|^2_{\infty,[-1,1]} + 1\Big)\, L^{-2\alpha}\, \|f\|^2_{\alpha},$$

i.e.,

$$\|e_L\|^2_{\mathrm{L}^2(\mathbb{R}^2)} = \mathcal{O}(L^{-2\alpha}) \quad \text{for} \quad L \longrightarrow \infty.$$

In summary, we can, under the above assumption, establish asymptotic L^2-error estimates for the FBP reconstruction with convergence rates as follows.

Theorem 5.5 (Rate of convergence)**.** *Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for $\alpha > 0$, and $W \in \mathcal{C}([-1,1])$ with $W(0) = 1$. Further, let Assumption 5.4 be satisfied. Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded above by*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \Big(c_{\alpha,W}^{-\alpha}\, \|1-W\|_{\infty,[-1,1]} + 1\Big)\, L^{-\alpha}\, \|f\|_{\alpha}, \tag{5.3}$$

i.e.,

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} = \mathcal{O}(L^{-\alpha}) \quad \textit{for} \quad L \longrightarrow \infty.$$

Note that the decay rate of the L^2-error in (5.3) is determined by the smoothness α of the target f. Further, for fixed target function f and bandwidth L, the obtained error estimate is again affine-linear with respect to $\|1-W\|_{\infty,[-1,1]}$, as in (4.1) and observed numerically in [1].

We remark that Assumption 5.4 is satisfied for a large class of window functions. For example, let the window function $W \in \mathcal{C}([-1,1])$ satisfy

$$W(S) = 1 \quad \forall\, S \in [-\varepsilon, \varepsilon]$$

for $\varepsilon > 0$ and

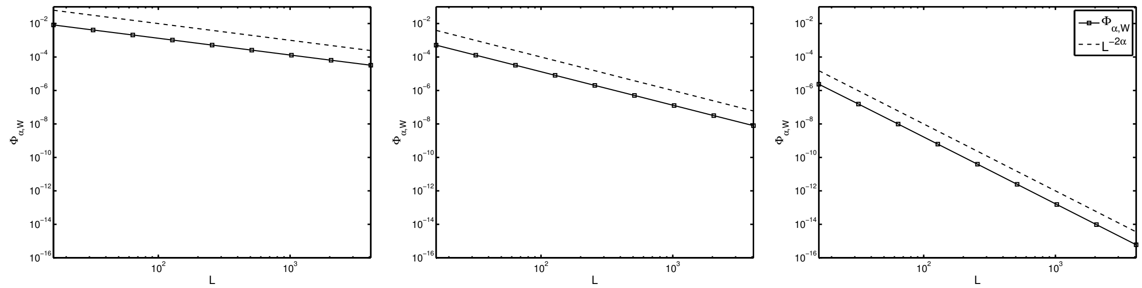
$$\exists\, R \in [0,1] :\; W(R) \neq 1.$$

Then, Assumption 5.4 is fulfilled with $c_{\alpha,W} = \varepsilon$.

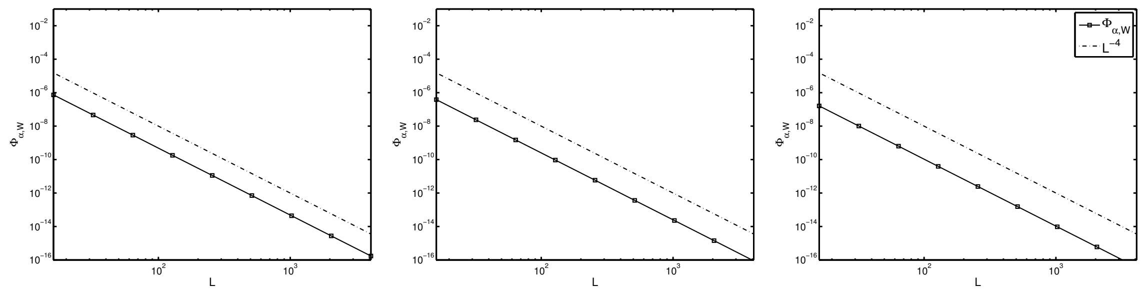
Numerical Observations. We investigate the behaviour of $S^*_{\alpha,W,L}$ and $\Phi_{\alpha,W}$ numerically for the following commonly used choices of the filter function $A_L(S) = |S|\, W(S/L)$:

Name	$W(S)$ for $\lvert S\rvert \leq 1$	Parameter
Shepp–Logan	$\mathrm{sinc}(\pi S/2)$	-
Cosine	$\cos(\pi S/2)$	-
Hamming	$\beta + (1-\beta)\cos(\pi S)$	$\beta \in [1/2, 1]$
Gaussian	$\exp\big(-(\pi S/\beta)^2\big)$	$\beta > 1$

Note that each of these window functions W is compactly supported with $\mathrm{supp}(W) = [-1,1]$.



(a) $\alpha = 0.5$ (b) $\alpha = 1$ (c) $\alpha = 2$



(d) $\alpha = 2.5$ (e) $\alpha = 3$ (f) $\alpha = 4$

FIGURE 1. Decay rate of $\Phi_{\alpha,W}$ for the Shepp–Logan filter.

In our numerical experiments, we calculated $S^*_{\alpha,W,L}$ and $\Phi_{\alpha,W}(L)$ as a function of the bandwidth $L > 0$ for the above mentioned window functions W and for different parameters $\alpha > 0$, reflecting the smoothness of the target function $f \in \mathrm{H}^\alpha(\mathbb{R}^2)$. Figure 1 shows the behaviour of $\Phi_{\alpha,W}$ in log-log scale for the Shepp–Logan filter and for smoothness parameters $\alpha \in \{0.5, 1, 2, 2.5, 3, 4\}$. For $\alpha \in \{0.5, 1, 2\}$ we observe that $\Phi_{\alpha,W}(L)$ behaves exactly as $L^{-2\alpha}$, see Figure 1(a)–(c), whereas for $\alpha \in \{2.5, 3, 4\}$ the behaviour of $\Phi_{\alpha,W}(L)$ corresponds to L^{-4}, see Figure 1(d)–(f). In the latter case, however, $\Phi_{\alpha,W}(L)$ decreases at increasing values $\alpha > 2$. We remark that the same behaviour was observed in our numerical experiments for the other window functions W mentioned above.

We summarize our numerical experiments (for all windows W listed above) as follows.

For $\alpha < 2$, we see that Assumption 5.4, i.e.,

$$\exists\, c_{\alpha,W} > 0\, \forall\, L > 0 :\ S^*_{\alpha,W,L} \geq c_{\alpha,W},$$

is fulfilled, where in particular,

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-2\alpha}) \quad \text{for} \quad L \longrightarrow \infty.$$

For $\alpha \geq 2$, we have

$$S^*_{\alpha,W,L} \longrightarrow 0 \quad \text{for} \quad L \longrightarrow \infty$$

and the convergence rate of $\Phi_{\alpha,W}$ stagnates at

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-4}) \quad \text{for} \quad L \longrightarrow \infty.$$

6. Error Analysis for $\mathcal{C}^2$-Windows

Note that all window functions W mentioned above are in $\mathcal{C}^2([-1,1])$. Therefore, in the following analysis we consider even window functions W with compact support in $[-1,1]$ that additionally satisfy $W \in \mathcal{C}^2([-1,1])$ and $W(0) = 1$.

Note that we require differentiability of the compactly supported window function W only on the interval $[-1,1]$. But we allow discontinuities of W at the boundary points of $[-1,1]$. As a first result, we obtain the following convergence rate.

Theorem 6.1 (Convergence rate of $\Phi_{\alpha,W}$ for $\mathcal{C}^2$-windows)**.** *Let the window function W satisfy $W \in \mathcal{C}^2([-1,1])$ with $W(0) = 1$. Moreover, let $\alpha > 0$. Then, we have*

$$\Phi_{\alpha,W}(L) \leq \begin{cases} C_\alpha \, \|W''\|^2_{\infty,[-1,1]} \, L^{-4} & \text{for } \alpha > 2 \wedge L \geq \frac{\sqrt{2}}{\sqrt{\alpha-2}} \\ \frac{1}{4} \, \|W''\|^2_{\infty,[-1,1]} \, L^{-2\alpha} & \text{for } \alpha \leq 2 \vee \Big(\alpha > 2 \wedge L < \frac{\sqrt{2}}{\sqrt{\alpha-2}}\Big) \end{cases} \qquad \forall \, L > 0,$$

i.e.,

$$\Phi_{\alpha,W}(L) = \mathcal{O}\Big(L^{-\min\{4,2\alpha\}}\Big) \qquad \text{for} \qquad L \longrightarrow \infty,$$

where the constant

$$C_\alpha = \frac{(\alpha-2)^{\alpha-2}}{\alpha^\alpha}$$

is strictly monotonically decreasing in $\alpha > 2$.

Proof. Since the window function W is assumed to be continuous on $[-1,1]$, we have

$$\Phi_{\alpha,W}(L) = \max_{S \in [-1,1]} \frac{(1 - W(S))^2}{(1 + L^2 S^2)^\alpha} = \max_{S \in [-1,1]} \Phi_{\alpha,W,L}(S).$$

Let $S \in [-1,1]$ be fixed. By assumption, W satisfies $W \in \mathcal{C}^2([-1,1])$ with $W(0) = 1$. Thus, we can apply Taylor's theorem and obtain

$$W(S) = W(0) + W'(0) \, S + \frac{1}{2} \, W''(\xi) \, S^2 = 1 + \frac{1}{2} \, W''(\xi) \, S^2$$

for some ξ between 0 and S, where we use that the window W is even and, consequently, $W'(0) = 0$. This leads to

$$\Phi_{\alpha,W,L}(S) = \frac{(W''(\xi))^2}{4} \, \frac{S^4}{(1 + L^2 \, S^2)^\alpha} \leq \frac{\|W''\|^2_{\infty,[-1,1]}}{4} \, \frac{S^4}{(1 + L^2 \, S^2)^\alpha}.$$

Hence,

$$\Phi_{\alpha,W}(L) \leq \frac{\|W''\|^2_{\infty,[-1,1]}}{4} \max_{S \in [-1,1]} \frac{S^4}{(1 + L^2 \, S^2)^\alpha} = \frac{\|W''\|^2_{\infty,[-1,1]}}{4} \max_{S \in [-1,1]} \phi_{\alpha,L}(S).$$

We now need to analyse the function

$$\phi_{\alpha,L}(S) = \frac{S^4}{(1 + L^2 \, S^2)^\alpha} \qquad \text{for } S \in [-1,1],$$

which is independent of the window function W. Since $\phi_{\alpha,L}$ is an even function, we have

$$\max_{S \in [-1,1]} \phi_{\alpha,L}(S) = \max_{S \in [0,1]} \phi_{\alpha,L}(S)$$

and so it suffices to consider $S \in [0,1]$. A necessary condition for a maximum of $\phi_{\alpha,L}$ on $(0,1)$ is

$$\phi'_{\alpha,L}(S) = 0.$$

From the first derivative

$$\phi'_{\alpha,L}(S) = \frac{2\,S^3\left(2 + (2-\alpha)L^2\,S^2\right)}{\left(1 + L^2\,S^2\right)^{\alpha+1}}$$

it follows that $\phi'_{\alpha,L}$ can vanish only for $S = 0$ or for $(\alpha - 2)L^2\,S^2 = 2$.

Now since $\phi_{\alpha,L}(0) = 0$ and $\phi_{\alpha,L}(S) > 0$, for all $S > 0$, it follows that $S = 0$ is the unique global minimizer of $\phi_{\alpha,L}$ on $[0, 1]$.

<u>Case 1:</u> For $0 \leq \alpha \leq 2$ the equation

$$(\alpha - 2)L^2\,S^2 = 2$$

has no solution in $[0, 1]$ and, moreover,

$$\phi'_{\alpha,L}(S) > 0 \quad \forall\, S \in (0, 1].$$

This means that $\phi_{\alpha,L}$ is strictly monotonically increasing on $(0, 1]$ and, thus, it is maximal on $[0, 1]$ for $S^* = 1$, i.e.,

$$\max_{S\in[0,1]} \phi_{\alpha,L}(S) = \phi_{\alpha,L}(1) = \frac{1}{\left(1 + L^2\right)^{\alpha}} \leq L^{-2\alpha}.$$

<u>Case 2:</u> For $\alpha > 2$ the unique positive solution of the equation

$$(\alpha - 2)L^2\,S^2 = 2$$

is given by

$$S^* = \frac{\sqrt{2}}{L\sqrt{\alpha - 2}},$$

where

$$S^* \in [0, 1] \qquad \Longleftrightarrow \qquad L \geq \frac{\sqrt{2}}{\sqrt{\alpha - 2}}.$$

For convenience, we define the function $g_{\alpha,L} : \mathbb{R} \longrightarrow \mathbb{R}$ via

$$g_{\alpha,L}(S) = 2 + (2 - \alpha)L^2\,S^2.$$

Then, $g_{\alpha,L}$ is a down open parabola with vertex in $S = 0$ and we obtain

$$g_{\alpha,L}(S_1) > g_{\alpha,L}(S_2) \quad \forall\, 0 \leq S_1 < S_2.$$

In particular, we have

$$g_{\alpha,L}(S_2) < g_{\alpha,L}(S^*) = 0 < g_{\alpha,L}(S_1) \quad \forall\, 0 < S_1 < S^* < S_2$$

and, consequently,

$$\phi'_{\alpha,L}(S_2) < \phi'_{\alpha,L}(S^*) = 0 < \phi'_{\alpha,L}(S_1) \quad \forall\, 0 < S_1 < S^* < S_2.$$

Thus, $\phi_{\alpha,L}$ is strictly monotonically increasing on $(0, S^*)$ and strictly monotonically decreasing on (S^*, ∞). Therefore, S^* is the unique maximizer of $\phi_{\alpha,L}$ and it follows that

$$\arg\max_{S\in[0,1]} \phi_{\alpha,L}(S) = \begin{cases} 1 & \text{for } L < \frac{\sqrt{2}}{\sqrt{\alpha-2}} \\ \frac{\sqrt{2}}{L\sqrt{\alpha-2}} & \text{for } L \geq \frac{\sqrt{2}}{\sqrt{\alpha-2}}. \end{cases}$$

Since

$$\phi_{\alpha,L}(S^*) = \frac{\left(\frac{\sqrt{2}}{L\sqrt{\alpha-2}}\right)^4}{\left(1 + L^2\left(\frac{\sqrt{2}}{L\sqrt{\alpha-2}}\right)^2\right)^{\alpha}} = 4\,\frac{(\alpha-2)^{2-\alpha}}{\alpha^{\alpha}}\,L^{-4}$$

we finally obtain (for $\alpha > 2$)

$$\max_{S\in[0,1]} \phi_{\alpha,L}(S) = \begin{cases} \phi_{\alpha,L}(1) & \text{for } L < \frac{\sqrt{2}}{\sqrt{\alpha-2}} \\ \phi_{\alpha,L}(S^*) & \text{for } L \geq \frac{\sqrt{2}}{\sqrt{\alpha-2}} \end{cases} \leq \begin{cases} L^{-2\alpha} & \text{for } L < \frac{\sqrt{2}}{\sqrt{\alpha-2}} \\ 4\,\frac{(\alpha-2)^{2-\alpha}}{\alpha^\alpha}\,L^{-4} & \text{for } L \geq \frac{\sqrt{2}}{\sqrt{\alpha-2}}. \end{cases}$$

Combining our results yields

$$\begin{aligned} \Phi_{\alpha,W}(L) &\leq \frac{1}{4}\,\|W''\|^2_{\infty,[-1,1]} \max_{S\in[0,1]} \phi_{\alpha,L}(S) \\ &\leq \frac{1}{4}\,\|W''\|^2_{\infty,[-1,1]} \begin{cases} 4\,\frac{(\alpha-2)^{2-\alpha}}{\alpha^\alpha}\,L^{-4} & \text{for } \alpha > 2 \wedge L \geq \frac{\sqrt{2}}{\sqrt{\alpha-2}} \\ L^{-2\alpha} & \text{for } \alpha \leq 2 \vee \left(\alpha > 2 \wedge L < \frac{\sqrt{2}}{\sqrt{\alpha-2}}\right) \end{cases} \\ &= \begin{cases} \frac{(\alpha-2)^{2-\alpha}}{\alpha^\alpha}\,\|W''\|^2_{\infty,[-1,1]}\,L^{-4} & \text{for } \alpha > 2 \wedge L \geq \frac{\sqrt{2}}{\sqrt{\alpha-2}} \\ \frac{1}{4}\,\|W''\|^2_{\infty,[-1,1]}\,L^{-2\alpha} & \text{for } \alpha \leq 2 \vee \left(\alpha > 2 \wedge L < \frac{\sqrt{2}}{\sqrt{\alpha-2}}\right), \end{cases} \end{aligned}$$

as stated.

Let us finally regard the constant

$$C_\alpha = C(\alpha) = \frac{(\alpha-2)^{\alpha-2}}{\alpha^\alpha}$$

as a function of $\alpha > 2$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\,C(\alpha) = \frac{(\alpha-2)^{\alpha-2}}{\alpha^\alpha}\,\log\!\left(1 - \frac{2}{\alpha}\right) < 0 \quad \forall\,\alpha > 2$$

and, consequently, C_α is strictly monotonically decreasing in $\alpha > 2$. □

We remark that the results of Theorem 6.1 comply with our numerical observations from the previous section. We have in particular observed saturation of the convergence rate of $\Phi_{\alpha,W}$ for $\alpha > 2$ at

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-4}) \quad \text{for} \quad L \longrightarrow \infty$$

through our numerical experiments. Therefore, our numerical results show that the proven order of convergence for $\Phi_{\alpha,W}$ is optimal for $\mathcal{C}^2$-windows.

By combining Theorems 5.1 and 6.1, we finally get the following result for the convergence order of FBP reconstruction with $\mathcal{C}^2$-windows.

Corollary 6.2 (L^2-error estimate for $\mathcal{C}^2$-windows)**.** *For $\alpha > 0$ let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$. Moreover, let $W \in \mathcal{C}^2([-1,1])$ with $W(0) = 1$. Then, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded above by*

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \begin{cases} \left(\frac{c_{\alpha,2}}{2}\,\|W''\|_{\infty,[-1,1]}\,L^{-2} + L^{-\alpha}\right)\|f\|_\alpha & \text{for } \alpha > 2 \wedge L \geq L^* \\ \left(\frac{1}{2}\,\|W''\|_{\infty,[-1,1]}\,L^{-\alpha} + L^{-\alpha}\right)\|f\|_\alpha & \text{for } \alpha \leq 2 \vee (\alpha > 2 \wedge L < L^*) \end{cases}$$

with the critical bandwidth $L^* = \frac{\sqrt{2}}{\sqrt{\alpha-2}}$, for $\alpha > 2$. Moreover, the constant

$$c_{\alpha,2} = \frac{2}{\alpha-2}\left(\frac{\alpha-2}{\alpha}\right)^{\alpha/2}$$

is strictly monotonically decreasing in $\alpha > 2$. In particular,

$$\|e_L\|_{L^2(\mathbb{R}^2)} \leq \Big(c\, \|W''\|_{\infty,[-1,1]}\, L^{-\min\{2,\alpha\}} + L^{-\alpha}\Big) \|f\|_\alpha = \mathcal{O}\Big(L^{-\min\{2,\alpha\}}\Big)\,.$$

We close this section by the following two remarks.

Firstly, note that the bound on the inherent FBP reconstruction error in Corollary 6.2 is affine-linear with respect to $\|W''\|_{\infty,[-1,1]}$. Therefore, the quantity in the upper bound can be used to evaluate the approximation quality of the chosen $\mathcal{C}^2$-window function W.

Secondly, for $\alpha \leq 2$ the convergence order of the approximate reconstruction f_L is given by the smoothness of the target function f. But for $\alpha > 2$ the convergence rate of the error bound saturates at $\mathcal{O}(L^{-2})$. Nevertheless, the FBP reconstruction error continues to decrease at increasing $\alpha > 2$, since the involved constant $c_{\alpha,2}$ is strictly monotonically decreasing in $\alpha > 2$. This matches our perceptions, as the approximation error should be smaller for target functions of higher regularity.

7. Error Analysis for $\mathcal{C}^k$-Windows

In this section, we generalize our results from the previous section to $\mathcal{C}^k$-windows whose first $k-1$ derivatives vanish at the origin. Therefore, we now consider even window functions W with compact support in $[-1,1]$ that additionally satisfy $W \in \mathcal{C}^k([-1,1])$ for some $k \geq 2$ and

$$W(0) = 1 \quad \text{and} \quad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

According to Theorem 5.2, $\Phi_{\alpha,W}(L)$ tends to zero for $L \to \infty$. In Theorem 6.1 we obtained convergence rates for $\Phi_{\alpha,W}$ with $\mathcal{C}^2$-windows W. We can prove convergence rates for $\mathcal{C}^k$-windows by following along the lines of the presented proofs for $k = 2$, see Theorem 6.1 and Corollary 6.2. We formulate our results for $k \geq 2$ as follows.

Theorem 7.1 (Convergence rate of $\Phi_{\alpha,W}$ for $\mathcal{C}^k$-windows)**.** *Let the window function W satisfy $W \in \mathcal{C}^k([-1,1])$, for $k \geq 2$, with*

$$W(0) = 1 \quad \textit{and} \quad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

Moreover, let $\alpha > 0$. Then, $\Phi_{\alpha,W}(L)$ can be bounded above by

$$\Phi_{\alpha,W}(L) \leq \begin{cases} \frac{c_{\alpha,k}^2}{(k!)^2}\, \|W^{(k)}\|^2_{\infty,[-1,1]}\, L^{-2k} & \text{for } \alpha > k \wedge L \geq L^* \\ \frac{1}{(k!)^2}\, \|W^{(k)}\|^2_{\infty,[-1,1]}\, L^{-2\alpha} & \text{for } \alpha \leq k \vee (\alpha > k \wedge L < L^*) \end{cases}$$

with the critical bandwidth $L^* = \frac{\sqrt{k}}{\sqrt{\alpha-k}}$, for $\alpha > k$, and the strictly increasing constant

$$c_{\alpha,k} = \Big(\frac{k}{\alpha-k}\Big)^{k/2} \Big(\frac{\alpha-k}{\alpha}\Big)^{\alpha/2} \qquad \text{for } \alpha > k.$$

In particular,

$$\Phi_{\alpha,W}(L) = \mathcal{O}\Big(L^{-2\min\{k,\alpha\}}\Big) \qquad \text{for} \quad L \longrightarrow \infty.$$

Combining Theorems 5.1 and 7.1, we obtain the following result concerning the convergence order of the FBP reconstruction with $\mathcal{C}^k$-windows.

Corollary 7.2 (L^2-error estimate for $\mathcal{C}^k$-windows). *For $\alpha > 0$ let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$. Moreoever, let $W \in \mathcal{C}^k([-1,1])$, for $k \geq 2$, with*

$$W(0) = 1 \quad \text{and} \quad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

Then, the L^2-norm of the inherent FBP reconstruction error $e_L = f - f_L$ is bounded above by

$$\|e_L\|_{\mathrm{L}^2} \leq \begin{cases} \left(\frac{c_{\alpha,k}}{k!}\, \|W^{(k)}\|_{\infty,[-1,1]}\, L^{-k} + L^{-\alpha}\right) \|f\|_\alpha & \text{for } \alpha > k \wedge L \geq L^* \\ \left(\frac{1}{k!}\, \|W^{(k)}\|_{\infty,[-1,1]}\, L^{-\alpha} + L^{-\alpha}\right) \|f\|_\alpha & \text{for } \alpha \leq k \vee (\alpha > k \wedge L < L^*)\,. \end{cases}$$

In particular,

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \Big(c\, \|W^{(k)}\|_{\infty,[-1,1]}\, L^{-\min\{k,\alpha\}} + L^{-\alpha}\Big)\, \|f\|_\alpha = \mathcal{O}\Big(L^{-\min\{k,\alpha\}}\Big)\,.$$

Note that our concluding remarks after Corollary 6.2 concerning the approximation order of the FBP reconstruction f_L continue to apply in the situation of $\mathcal{C}^k$-windows W. Indeed, the convergence order in Corollary 7.2, for $\alpha \leq k$, is determined by the smoothness of the target function f, whereas for $\alpha > k$ the convergence rate saturates at $\mathcal{O}(L^{-k})$. But in this case the error bound decreases at increasing α, since the involved constant $c_{\alpha,k}$ is strictly monotonically decreasing in $\alpha > k$. Thus, a smoother target function allows for a better approximation, as expected. Nevertheless, the attainable convergence rate is limited by the differentiability order k of the filter's $\mathcal{C}^k$-window W.

Finally, note that the bound on the inherent FBP reconstruction error in Corollary 7.2 is affine-linear with respect to $\|W^{(k)}\|_{\infty,[-1,1]}$ and this quantity can be used to evaluate the approximation quality of the chosen $\mathcal{C}^k$-window function W.

Numerical Experiments. We investigate the behaviour of $\Phi_{\alpha,W}$ numerically for the generalized Gaussian filter $A_L(S) = |S|\, W(S/L)$ with the window function

$$W(S) = \exp\left(-\left(\frac{\pi S}{\beta}\right)^k\right) \qquad \text{for } S \in [-1,1]$$

for an even $k \in \mathbb{N}_{\geq 2}$ and $\beta > 1$. In this case, $W \in \mathcal{C}^k([-1,1])$ is even and compactly supported with $\mathrm{supp}(W) \subseteq [-1,1]$. Moreover,

$$W(0) = 1 \quad \text{and} \quad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1 \quad \text{and} \quad W^{(k)}(0) = -k! \left(\frac{\pi}{\beta}\right)^k \neq 0.$$

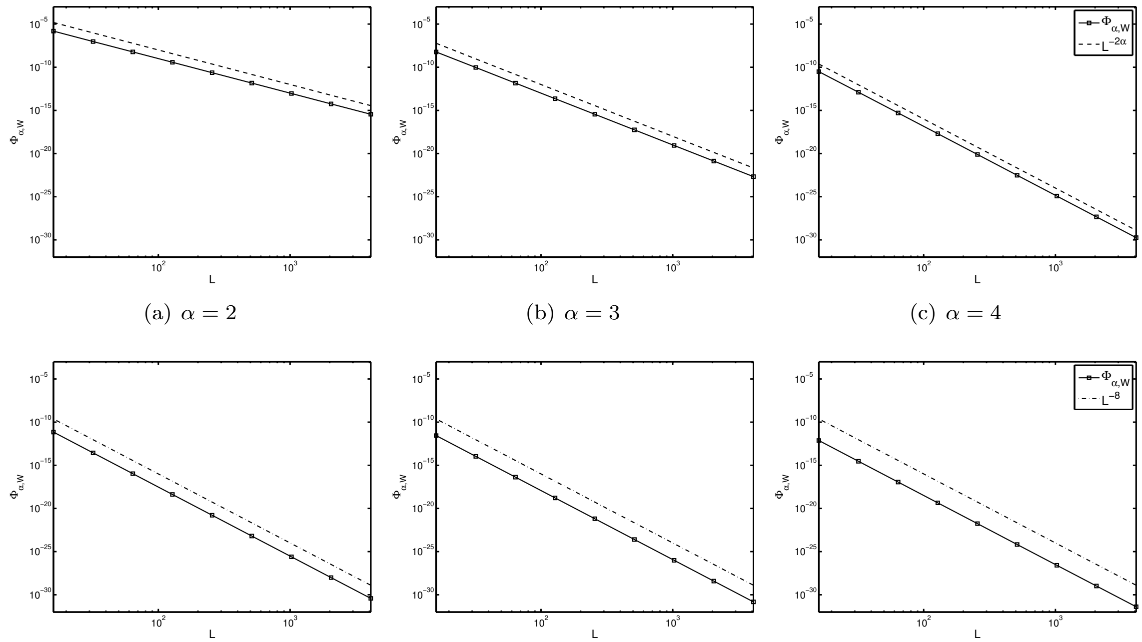
In our numerical experiments, we evaluated $\Phi_{\alpha,W}(L)$ as a function of the bandwidth $L > 0$ for the Gaussian's window W, using various combinations of parameters $k \in \mathbb{N}_{\geq 2}$, $\beta > 1$, and $\alpha > 0$. Figure 2 shows the behaviour of $\Phi_{\alpha,W}$ in log-log scale for the generalized Gaussian filter with $k = 4$ and $\beta = 4$, for the smoothness parameters $\alpha \in \{2, 3, 4, 4.5, 5, 6\}$. For $\alpha \in \{2, 3, 4\}$ we observe that $\Phi_{\alpha,W}(L)$ behaves as $L^{-2\alpha}$, see Figure 2(a)–(c), whereas for $\alpha \in \{4.5, 5, 6\}$ the behaviour of $\Phi_{\alpha,W}(L)$ corresponds to L^{-8}, see Figure 2(d)–(f). But $\Phi_{\alpha,W}(L)$ continues to decrease at increasing $\alpha > k$.

We can summarize the results of our numerical experiments as follows. For $\alpha < k$, we observe

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-2\alpha}) \quad \text{for} \quad L \longrightarrow \infty.$$

For $\alpha \geq k$, the convergence rate of $\Phi_{\alpha,W}$ saturates at

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-2k}) \quad \text{for} \quad L \longrightarrow \infty.$$



(a) $\alpha = 2$ (b) $\alpha = 3$ (c) $\alpha = 4$

(d) $\alpha = 4.5$ (e) $\alpha = 5$ (f) $\alpha = 6$

FIGURE 2. Decay rate of $\Phi_{\alpha,W}$ for the generalized Gaussian filter with $k = 4$, $\beta = 4$.

Note that the results of Theorem 7.1 entirely comply with our numerical observations (for the generalized Gaussian filters). So have we, in particular, observed the saturation of the convergence rate of $\Phi_{\alpha,W}$ for $\alpha > k$ at

$$\Phi_{\alpha,W}(L) = \mathcal{O}(L^{-2k}) \quad \text{for} \quad L \longrightarrow \infty.$$

Our numerical results show that the proven convergence order of $\Phi_{\alpha,W}$ is optimal for $\mathcal{C}^k$-windows.

Asymptotic Error Estimates. In this subsection, we take a different approach to prove asymptotic error estimates for the proposed FBP reconstruction method with window functions which are k-times differentiable only at the origin. To this end, we now consider an even window function $W \in \mathrm{L}^\infty(\mathbb{R})$, with compact support on $[-1, 1]$. Moreover, W is required to have k derivatives at zero, for some $k \geq 2$, with

$$W(0) = 1 \quad \text{and} \quad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

As in the previous sections, we consider target functions $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for some $\alpha > 0$. For the sake of brevity, we again set $r(x,y) = \sqrt{x^2+y^2}$ for $(x,y) \in \mathbb{R}^2$.

Recall the representation of the FBP reconstruction error $e_L = f - f_L$ with respect to the L^2-norm in (4.2), by the sum of two integrals, I_1 in (4.3) and I_2 in (4.4), where integral I_2 can be bounded above by (5.1), i.e.,

$$I_2 \leq L^{-2\alpha}\, \|f\|_\alpha^2.$$

As regards integral I_1, we have

$$I_1 = \frac{1}{4\pi^2} \int_{r(x,y)\leq L} |1 - W_L(r(x,y))|^2 \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y)$$
$$= \frac{1}{4\pi^2} \int_{r(x,y)\leq L} \left|1 - W\left(\frac{r(x,y)}{L}\right)\right|^2 \, |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y).$$

Because $W : \mathbb{R} \longrightarrow \mathbb{R}$ is k-times differentiable at zero, we can apply Taylor's theorem and, thus, there exists a function $h_k : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying

$$W(S) = \sum_{j=0}^{k} \frac{W^{(j)}(0)}{j!} \, S^j + h_k(S) \, S^k \quad \forall \, S \in \mathbb{R}$$

and

$$\lim_{S\to 0} h_k(S) = 0.$$

By assumption, W satisfies

$$W(0) = 1 \quad \text{ and } \quad W^{(j)}(0) = 0 \quad \forall \, 1 \leq j \leq k-1.$$

Hence, for $(x,y) \in \mathbb{R}^2$ and $L > 0$ follows that

$$1 - W\left(\frac{r(x,y)}{L}\right) = -\left(\frac{W^{(k)}(0)}{k!} \, \left(\frac{r(x,y)}{L}\right)^k + h_k\left(\frac{r(x,y)}{L}\right) \, \left(\frac{r(x,y)}{L}\right)^k\right),$$

so that we obtain the representation

$$I_1 = \frac{1}{4\pi^2} \int_{r(x,y)\leq L} \left(\frac{W^{(k)}(0)}{k!} + h_k\left(\frac{r(x,y)}{L}\right)\right)^2 \left(\frac{r(x,y)}{L}\right)^{2k} |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y).$$

For convenience, we define

$$\phi^*_{\alpha,L,k} = \max_{r(x,y)\leq L} \frac{\left(\frac{r(x,y)}{L}\right)^{2k}}{(1 + r(x,y)^2)^\alpha} = \max_{S\in[0,1]} \frac{S^{2k}}{(1 + L^2 \, S^2)^\alpha}.$$

Then, I_1 can be bounded above by

$$I_1 \leq \phi^*_{\alpha,L,k} \, \frac{1}{4\pi^2} \int_{r(x,y)\leq L} \left(\frac{W^{(k)}(0)}{k!} + h_k\left(\frac{r(x,y)}{L}\right)\right)^2 \left(1 + r(x,y)^2\right)^\alpha |\mathcal{F}f(x,y)|^2 \, \mathrm{d}(x,y).$$

We now regard the integral

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left(h_k\left(\frac{r(x,y)}{L}\right)\right)^2 \left(1 + r(x,y)^2\right)^\alpha |\mathcal{F}f(x,y)|^2 \, \mathrm{d}x \, \mathrm{d}y.$$

For $S \neq 0$, the function h_k can be written as

$$h_k(S) = (W(S) - 1) \, S^{-k} - \frac{W^{(k)}(0)}{k!}.$$

Since the window function W is compactly supported in $[-1, 1]$, we obtain

$$h_k(S) = -S^{-k} - \frac{W^{(k)}(0)}{k!} \quad \forall \, |S| > 1,$$

which implies

$$h_k(S) \longrightarrow -\frac{W^{(k)}(0)}{k!} \quad \text{for} \quad S \longrightarrow \pm\infty.$$

From $W \in \mathrm{L}^\infty(\mathbb{R})$ and

$$h_k(S) \longrightarrow 0 \quad \text{for} \quad S \longrightarrow 0$$

it follows that h_k is bounded on $\mathbb{R}$, so that there exists some constant $M > 0$ satisfying

$$\left| h_k\left(\frac{r(x,y)}{L}\right)\right|^2 \leq M \quad \forall\,(x,y) \in \mathbb{R}^2,\, L > 0.$$

Hence, for all $L > 0$, the integrand

$$h_{k,L}(x,y) = \left(h_k\left(\frac{r(x,y)}{L}\right)\right)^2 \left(1 + r(x,y)^2\right)^\alpha |\mathcal{F}f(x,y)|^2$$

is bounded on $\mathbb{R}^2$ by the function

$$\Phi(x,y) = M\, \left(1 + r(x,y)^2\right)^\alpha |\mathcal{F}f(x,y)|^2,$$

which is integrable over $\mathbb{R}^2$ due to the assumption $f \in \mathrm{H}^\alpha(\mathbb{R}^2)$. Moreover, we have

$$h_k\left(\frac{r(x,y)}{L}\right) \longrightarrow 0 \quad \text{for} \quad \frac{r(x,y)}{L} \longrightarrow 0,$$

which implies that, for any $(x,y) \in \mathbb{R}^2$, $h_{k,L}(x,y)$ tends to zero as L goes to ∞. Thus, we can apply Lebesgue's theorem on dominated convergence to get

$$\lim_{L\to\infty} \int_{\mathbb{R}}\int_{\mathbb{R}} \left(h_k\left(\frac{r(x,y)}{L}\right)\right)^2 \left(1 + r(x,y)^2\right)^\alpha |\mathcal{F}f(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y = 0,$$

i.e.,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \left(h_k\left(\frac{r(x,y)}{L}\right)\right)^2 \left(1 + r(x,y)^2\right)^\alpha |\mathcal{F}f(x,y)|^2 \,\mathrm{d}x\,\mathrm{d}y = o(1) \quad \text{for} \quad L \longrightarrow \infty.$$

This leads us to the estimate

$$\begin{aligned}
I_1 &\leq \phi^*_{\alpha,L,k}\, \frac{1}{4\pi^2} \int_{r(x,y)\leq L} \underbrace{\left(\frac{W^{(k)}(0)}{k!} + h_k\left(\frac{r(x,y)}{L}\right)\right)^2}_{\leq 2\left(\frac{W^{(k)}(0)}{k!}\right)^2 + 2\left(h_k\left(\frac{r(x,y)}{L}\right)\right)^2} \left(1 + r(x,y)^2\right)^\alpha |\mathcal{F}f(x,y)|^2 \,\mathrm{d}(x,y) \\
&\leq 2\,\phi^*_{\alpha,L,k}\, \frac{1}{4\pi^2} \int_{r(x,y)\leq L} \left(\frac{W^{(k)}(0)}{k!}\right)^2 \left(1 + r(x,y)^2\right)^\alpha |\mathcal{F}f(x,y)|^2 \,\mathrm{d}(x,y) \\
&\quad + 2\,\phi^*_{\alpha,L,k}\, \frac{1}{4\pi^2} \int_{r(x,y)\leq L} \left(h_k\left(\frac{r(x,y)}{L}\right)\right)^2 \left(1 + r(x,y)^2\right)^\alpha |\mathcal{F}f(x,y)|^2 \,\mathrm{d}(x,y) \\
&\leq 2\,\phi^*_{\alpha,L,k} \left(\frac{W^{(k)}(0)}{k!}\right)^2 \|f\|_\alpha^2 + \phi^*_{\alpha,L,k}\, o(1).
\end{aligned}$$

Using the same technique as in the proof of Theorem 6.1, we can bound $\phi^*_{\alpha,L,k}$ by

$$\phi^*_{\alpha,L,k} \leq \begin{cases} \left(\frac{k}{\alpha-k}\right)^k \left(\frac{\alpha-k}{\alpha}\right)^\alpha L^{-2k} & \text{for } \alpha > k \wedge L \geq L^* \\ L^{-2\alpha} & \text{for } \alpha \leq k \vee (\alpha > k \wedge L < L^*) \end{cases} = \mathcal{O}\left(L^{-2\min\{k,\alpha\}}\right)$$

with the critical bandwidth $L^* = \frac{\sqrt{k}}{\sqrt{\alpha-k}}$ for $\alpha > k$. Thus, it follows that

$$I_1 \leq \begin{cases} \frac{2}{(k!)^2}\, c_{\alpha,k}^2\, |W^{(k)}(0)|^2\, L^{-2k}\, \|f\|_\alpha^2 + o\big(L^{-2k}\big) & \text{for } \alpha > k \wedge L \geq L^* \\ \frac{2}{(k!)^2}\, |W^{(k)}(0)|^2\, L^{-2\alpha}\, \|f\|_\alpha^2 + o\big(L^{-2\alpha}\big) & \text{for } \alpha \leq k \vee (\alpha > k \wedge L < L^*)\,, \end{cases}$$

where the constant

$$c_{\alpha,k} = \Big(\frac{k}{\alpha-k}\Big)^{k/2}\Big(\frac{\alpha-k}{\alpha}\Big)^{\alpha/2} \qquad \text{for } \alpha > k$$

is strictly monotonically decreasing in $\alpha > k$ (cf. Theorem 7.1).

By combining our derived bounds for the integrals I_1 and I_2, we finally get the L^2-error estimate

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)}^2 \leq \bigg(2\left(C_{\alpha,k}\,|W^{(k)}(0)|\right)^2 L^{-2\min\{k,\alpha\}} + L^{-2\alpha}\bigg)\,\|f\|_\alpha^2 + o\Big(L^{-2\min\{k,\alpha\}}\Big)\,.$$

In conclusion, we have proven the following error theorem for the FBP reconstruction method.

Theorem 7.3 (Asymptotic L^2-error estimate)**.** *For $\alpha > 0$ let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$. Moreover, let $W \in \mathrm{L}^\infty(\mathbb{R})$ be even, with* $\mathrm{supp}(W) \subseteq [-1,1]$*, and k-times differentiable at the origin, $k \geq 2$, with*

$$W(0) = 1 \quad \textit{and} \quad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

Then, for $\alpha \leq k$, the L^2-norm of the FBP reconstruction error $e_L = f - f_L$ is bounded above by

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(\frac{\sqrt{2}}{k!}\,|W^{(k)}(0)|\,L^{-\alpha} + L^{-\alpha}\right)\|f\|_\alpha + o(L^{-\alpha}). \tag{7.1}$$

If $\alpha > k$, the L^2-norm of e_L can be bounded above by

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \begin{cases} \Big(\frac{\sqrt{2}}{k!}\,c_{\alpha,k}|W^{(k)}(0)|\,L^{-k} + L^{-\alpha}\Big)\,\|f\|_\alpha + o(L^{-k}) & \textit{for } L \geq L^* \\ \Big(\frac{\sqrt{2}}{k!}\,|W^{(k)}(0)|\,L^{-\alpha} + L^{-\alpha}\Big)\,\|f\|_\alpha + o(L^{-\alpha}) & \textit{for } L < L^* \end{cases} \tag{7.2}$$

with the critical bandwidth $L^* = \frac{\sqrt{k}}{\sqrt{\alpha-k}}$ and the strictly monotonically decreasing constant

$$c_{\alpha,k} = \Big(\frac{k}{\alpha-k}\Big)^{k/2}\Big(\frac{\alpha-k}{\alpha}\Big)^{\alpha/2} \qquad \textit{for } \alpha > k.$$

In particular,

$$\|e_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \Big(c\,|W^{(k)}(0)|\,L^{-\min\{k,\alpha\}} + L^{-\alpha}\Big)\,\|f\|_\alpha + o\Big(L^{-\min\{k,\alpha\}}\Big)\,.$$

We wish to draw the following conclusions from Theorem 7.3.

Firstly, the *flatness* of the filter's window function W determines the convergence rate of the error bounds (7.1), (7.2) for the inherent FBP reconstruction error. Indeed, if W is k-times differentiable at the origin such that the first $k-1$ derivatives of W vanish at zero, then the convergence rate in (7.1) is given by the smoothness α of the target function f as long as $\alpha \leq k$. But for $\alpha > k$ the order of convergence in (7.2) saturates at $\mathcal{O}(L^{-k})$.

Secondly, the quantity $|W^{(k)}(0)|$, i.e., the k-th derivative of W at the origin, dominates the error bound in both (7.1) and (7.2). Therefore, the value $|W^{(k)}(0)|$ can be used as an indicator to predict the approximation quality of the proposed FBP reconstruction method.

To conclude our discussion, we finally consider the following special case. Let the window function W fulfil the assumptions of Theorem 7.3 with $k \geq 2$ and let the smoothness α of $f \in \mathrm{H}^\alpha(\mathbb{R}^2)$ satisfy

$$\alpha > k.$$

Then, the asymptotic L^2-error estimate of the FBP method reduces to

$$\|f - f_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \frac{\sqrt{2}}{k!}\, c_{\alpha,k}\, |W^{(k)}(0)|\, L^{-k}\, \|f\|_\alpha + o(L^{-k}).$$

Consequently, the intrinsic FBP reconstruction error is proportional to $|W^{(k)}(0)|$, if we neglect the higher order terms. For $k = 2$, this observation complies with the results of Munshi [13] and Munshi et al. [14, 15], where they assumed certain moment conditions on the convolution kernel K and differentiability of the target function f in a strict sense.

8. Convergence Rates for Noisy Data

We finally turn to the important case of noisy data. In fact, for many relevant applications, the Radon data $g = \mathcal{R}f \in \mathrm{L}^2(\mathbb{R} \times [0,\pi))$ is not known exactly, but only up to an error $\delta > 0$, so that we wish to reconstruct f from given noisy measurements $g^\delta \in \mathrm{L}^1(\mathbb{R} \times [0,\pi)) \cap \mathrm{L}^2(\mathbb{R} \times [0,\pi))$, where

$$\left\|g - g^\delta\right\|_{\mathrm{L}^2(\mathbb{R}\times[0,\pi))} \leq \delta.$$

Applying the approximate FBP formula (2.2) to the noisy data g^δ, this yields the reconstruction

$$f_L^\delta = \frac{1}{2}\,\mathcal{B}\big(q_L * g^\delta\big). \tag{8.1}$$

Using standard concepts from inverse problems and regularization theory, we see that the overall FBP reconstruction error

$$e_L^\delta = f - f_L^\delta \tag{8.2}$$

can be split into an *approximation error* term and a *data error* term,

$$e_L^\delta = \underbrace{f - f_L}_{\substack{\text{approximation}\\ \text{error}}} + \underbrace{f_L - f_L^\delta}_{\substack{\text{data}\\ \text{error}}}.$$

In the following of this section, we analyse the L^2-norm of the overall FBP reconstruction error e_L^δ in (8.2) with respect to the noise level δ as well as the filter's window function W and bandwidth L. To this end, we first show that the noisy FBP reconstruction f_L^δ in (8.1) also satisfies $f_L^\delta \in \mathrm{L}^2(\mathbb{R}^2)$. By the triangle inequality we have

$$\|e_L^\delta\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \|f - f_L\|_{\mathrm{L}^2(\mathbb{R}^2)} + \|f_L - f_L^\delta\|_{\mathrm{L}^2(\mathbb{R}^2)}.$$

Hence, we will estimate the data error (in Section 8.1) and the approximation error (in Section 8.2) separately. In preparation, we first need to collect a few relevant results concerning the Radon transform. Since the following results are well-known, we omit the proofs and refer to the literature instead. We first recall that for $f \in \mathrm{L}^1(\mathbb{R}^2)$ the Radon transform $\mathcal{R}f$ is in $\mathrm{L}^1(\mathbb{R} \times [0,\pi))$.

Lemma 8.1. *The Radon transform $\mathcal{R} : \mathrm{L}^1(\mathbb{R}^2) \longrightarrow \mathrm{L}^1(\mathbb{R} \times [0,\pi))$ is continuous. In particular, for $f \in \mathrm{L}^1(\mathbb{R}^2)$ we have*

$$\|\mathcal{R}f\|_{\mathrm{L}^1(\mathbb{R}\times[0,\pi))} \leq \pi\, \|f\|_{\mathrm{L}^1(\mathbb{R}^2)}.$$

Next we recall that the L^2-norm of $\mathcal{R}f$ is bounded for $f \in \mathrm{L}^2_c(\mathbb{R}^2)$, i.e., for f square integrable and compactly supported.

Lemma 8.2. *Let $f \in \mathrm{L}^2_c(\mathbb{R}^2)$ be supported in a compact set $K \subset \mathbb{R}^2$ with diameter*

$$\operatorname{diam}(K) = \sup\{\|(x - X, y - Y)\|_{\mathbb{R}^2} \mid (x, y), (X, Y) \in K\} < \infty.$$

Then, $\mathcal{R}f \in \mathrm{L}^2(\mathbb{R} \times [0, \pi))$, where

$$\|\mathcal{R}f\|^2_{\mathrm{L}^2(\mathbb{R}\times[0,\pi))} \leq \pi \operatorname{diam}(K) \, \|f\|^2_{\mathrm{L}^2(\mathbb{R}^2)}.$$

By Lemma 8.2 the Radon transform $\mathcal{R}$ is a densely defined unbounded linear operator from $\mathrm{L}^2(\mathbb{R}^2)$ to $\mathrm{L}^2(\mathbb{R} \times [0, \pi))$ with domain $\mathrm{L}^2_c(\mathbb{R}^2)$. Next we turn to the adjoint operator $\mathcal{R}^\#$ of $\mathcal{R}$.

Lemma 8.3 (see [26, Theorem 12.3])**.** *The adjoint operator $\mathcal{R}^\#$ of $\mathcal{R} : \mathrm{L}^2_c(\mathbb{R}^2) \longrightarrow \mathrm{L}^2(\mathbb{R} \times [0, \pi))$ is given by*

$$\mathcal{R}^\# g(x, y) = \int_0^\pi g(x \cos(\theta) + y \sin(\theta), \theta) \, \mathrm{d}\theta \qquad \text{for } (x, y) \in \mathbb{R}^2.$$

For every $g \in \mathrm{L}^2(\mathbb{R} \times [0, \pi))$, $\mathcal{R}^\# g$ is defined almost everywhere on $\mathbb{R}^2$ and satisfies

$$\mathcal{R}^\# g \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2).$$

Lemma 8.3 shows that, up to the constant $\frac{1}{\pi}$, the back projection operator $\mathcal{B}$ is the adjoint operator of the Radon transform $\mathcal{R}$, i.e.,

$$\mathcal{B} = \frac{1}{\pi} \mathcal{R}^\#.$$

In particular, for $g \in \mathrm{L}^2(\mathbb{R} \times [0, \pi))$ the function $\mathcal{B}g$ is defined almost everywhere on $\mathbb{R}^2$ and satisfies

$$\mathcal{B}g \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2).$$

Finally, recall the standard *Schwartz space*

$$\mathcal{S}(\mathbb{R}^2) = \{f \in \mathcal{C}^\infty(\mathbb{R}^2) \mid \forall \, \alpha, \beta \in \mathbb{N}_0^2 : \; |f|_{\alpha,\beta} < \infty\}$$

of all rapidly decaying $\mathcal{C}^\infty$-functions on $\mathbb{R}^2$, where

$$|f|_{\alpha,\beta} = \sup_{(x,y)\in\mathbb{R}^2} |(x, y)^\alpha \, \mathrm{D}^\beta f(x, y)| \qquad \text{for } \alpha, \beta \in \mathbb{N}_0^2.$$

Likewise, the Schwartz space $\mathcal{S}(\mathbb{R} \times [0, \pi))$ can also be defined on $\mathbb{R} \times [0, \pi)$, in which case, for any $f \equiv f(S, \theta) \in \mathcal{S}(\mathbb{R} \times [0, \pi))$, its rapid decay is only with respect to the radial variable $S \in \mathbb{R}$. The next lemma shows that the Radon transform of any $f \in \mathcal{S}(\mathbb{R}^2)$ lies in $\mathcal{S}(\mathbb{R} \times [0, \pi)) \subset \mathrm{L}^2(\mathbb{R} \times [0, \pi))$.

Lemma 8.4 (see [5, Theorem 4.1])**.** *The Radon transform $\mathcal{R} : \mathcal{S}(\mathbb{R}^2) \longrightarrow \mathcal{S}(\mathbb{R} \times [0, \pi))$ is continuous.*

Recall that the back projection operator $\mathcal{B}$ is (up to constant $1/\pi$) the dual operator of $\mathcal{R}$ by

$$(\mathcal{R}f, g)_{\mathrm{L}^2(\mathbb{R}\times[0,\pi))} = \pi \, (f, \mathcal{B}g)_{\mathrm{L}^2(\mathbb{R}^2)} \quad \forall \, f \in \mathcal{S}(\mathbb{R}^2), \, g \in \mathcal{S}(\mathbb{R} \times [0, \pi)).$$

Therefore, we conclude from Lemma 8.4 that $\mathcal{B}g$ is a tempered distribution on $\mathbb{R}^2$, $\mathcal{B}g \in \mathcal{S}'(\mathbb{R}^2)$, for all $g \in \mathcal{S}'(\mathbb{R} \times [0, \pi))$. Moreover, since $\mathrm{L}^2(\mathbb{R} \times [0, \pi)) \subset \mathcal{S}'(\mathbb{R} \times [0, \pi))$, we have

$$\mathcal{B}g \in \mathcal{S}'(\mathbb{R}^2) \quad \forall \, g \in \mathrm{L}^2(\mathbb{R} \times [0, \pi)). \tag{8.3}$$

8.1. **Analysis of the data error.** Now we analyse the data error $f_L - f_L^\delta$ in the L^2-norm. To this end, we first show that

$$R_L g = \frac{1}{2}\,\mathcal{B}\big(q_L * g\big)$$

defines a continuous linear regularization operator

$$R_L : \mathrm{L}^1(\mathbb{R}\times[0,\pi)) \cap \mathrm{L}^2(\mathbb{R}\times[0,\pi)) \longrightarrow \mathrm{L}^2(\mathbb{R}^2).$$

Theorem 8.5. *Let $g \in \mathrm{L}^1(\mathbb{R}\times[0,\pi)) \cap \mathrm{L}^2(\mathbb{R}\times[0,\pi))$. Then, we have $R_L g \in \mathrm{L}^2(\mathbb{R}^2)$, where*

$$\|R_L g\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \frac{1}{\sqrt{2\pi}} \left(\sup_{S\in[-1,1]} |S|\,|W(S)|^2\right)^{1/2} L^{1/2}\,\|g\|_{\mathrm{L}^2(\mathbb{R}\times[0,\pi))}.$$

Proof. Since $A_L \in \mathrm{L}^1(\mathbb{R}) \cap \mathrm{L}^2(\mathbb{R})$, for all $L > 0$, the band-limited function q_L is well-defined on $\mathbb{R}\times[0,\pi)$ and we have $q_L \in \mathrm{L}^2(\mathbb{R}\times[0,\pi))$. Therefore, for all $\theta \in [0,\pi)$ the Fourier inversion formula

$$A_L(S) = \mathcal{F}(\mathcal{F}^{-1}A_L)(S) = \mathcal{F}q_L(S,\theta)$$

holds in the L^2-sense, in particular for almost all $S \in \mathbb{R}$. Since $g \in \mathrm{L}^1(\mathbb{R}\times[0,\pi))$, we obtain

$$A_L(S)\,\mathcal{F}g(S,\theta) = \mathcal{F}(q_L * g)(S,\theta) \quad \text{for almost all } S \in \mathbb{R}$$

by the Fourier convolution theorem. Moreover, Young's inequality yields $(q_L * g)(\cdot,\theta) \in \mathrm{L}^2(\mathbb{R})$, for any $\theta \in [0,\pi)$. This in combination with the Fourier inversion formula (in the L^2-sense) gives

$$(q_L * g)(S,\theta) = \mathcal{F}^{-1}[A_L(S)\,\mathcal{F}g(S,\theta)] \quad \text{for almost all } S \in \mathbb{R}.$$

In particular, we have $(q_L * g) \in \mathrm{L}^2(\mathbb{R}\times[0,\pi))$. Therefore,

$$R_L g = \frac{1}{2}\,\mathcal{B}\big(q_L * g\big)$$

is well-defined almost everywhere on $\mathbb{R}^2$ and satisfies $R_L g \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^2)$, due to Lemma 8.3.

On the other hand, we have $R_L g \in \mathcal{S}'(\mathbb{R}^2)$ by (8.3). This allows us to determine the (distributional) Fourier transform of $R_L g$, as being defined via the duality relation

$$\langle \mathcal{F}(R_L g), w\rangle = \langle R_L g, \mathcal{F}w\rangle = \frac{1}{2}\,(\mathcal{B}(q_L * g), \mathcal{F}w)_{\mathrm{L}^2(\mathbb{R}^2)} \quad \forall\, w \in \mathcal{S}(\mathbb{R}^2).$$

Now for any Schwartz function $w \in \mathcal{S}(\mathbb{R}^2)$, we have

$$\langle R_L g, \mathcal{F}w\rangle = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\pi (q_L * g)(x\cos(\theta) + y\sin(\theta),\theta)\,\mathrm{d}\theta\,\mathcal{F}w(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

by the definition of the back projection $\mathcal{B}$. From this, and by using the parameter transformation

$$x = t\cos(\theta) - s\sin(\theta) \quad \text{and} \quad y = t\sin(\theta) + s\cos(\theta),$$

we obtain

$$\begin{aligned}
\langle R_L g, \mathcal{F}w\rangle &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^\pi (q_L * g)(t,\theta)\,\mathcal{F}w(t\cos(\theta) - s\sin(\theta), t\sin(\theta) + s\cos(\theta))\,\mathrm{d}\theta\,\mathrm{d}t\,\mathrm{d}s \\
&= \frac{1}{2\pi}\int_0^\pi\int_{\mathbb{R}} (q_L * g)(t,\theta)\,\mathcal{R}(\mathcal{F}w)(t,\theta)\,\mathrm{d}t\,\mathrm{d}\theta
\end{aligned}$$

by Fubini's theorem and by the definition of the Radon transform $\mathcal{R}$. Now Parseval's identity gives

$$\int_{\mathbb{R}} \mathcal{F}^{-1}f(x)\,h(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)\,\mathcal{F}^{-1}h(x)\,\mathrm{d}x \qquad \forall f,h \in \mathrm{L}^1(\mathbb{R}).$$

Recall that, for any $\theta \in [0, \pi)$, we have

$$(q_L * g)(t, \theta) = \mathcal{F}^{-1}[A_L(t)\, \mathcal{F}g(t, \theta)] \quad \text{for almost all } t \in \mathbb{R},$$

where $A_L(\cdot)\, \mathcal{F}g(\cdot, \theta) \in \mathrm{L}^1(\mathbb{R})$, since $A_L \in \mathrm{L}^2(\mathbb{R})$ and $g \in \mathrm{L}^2(\mathbb{R} \times [0, \pi))$. Further recall that the two operators $\mathcal{F} : \mathcal{S}(\mathbb{R}^2) \longrightarrow \mathcal{S}(\mathbb{R}^2)$ and $\mathcal{R} : \mathrm{L}^1(\mathbb{R}^2) \longrightarrow \mathrm{L}^1(\mathbb{R} \times [0, \pi))$ are continuous, respectively. Moreover, since $\mathcal{S}(\mathbb{R}^2) \subset \mathrm{L}^1(\mathbb{R}^2)$, we have

$$\mathcal{R}(\mathcal{F}w)(\cdot, \theta) \in \mathrm{L}^1(\mathbb{R}) \quad \forall\, w \in S(\mathbb{R}^2)$$

for any $\theta \in [0, \pi)$. Therefore, the application of Parseval's identity yields

$$\langle R_L g, \mathcal{F}w \rangle = \frac{1}{2\pi} \int_0^\pi \int_{\mathbb{R}} A_L(t)\, \mathcal{F}g(t, \theta)\, \mathcal{F}^{-1}(\mathcal{R}(\mathcal{F}w))(t, \theta)\, \mathrm{d}t\, \mathrm{d}\theta.$$

To continue our analysis, we note that the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are related via

$$\mathcal{F}^{-1} f = (2\pi)^{-n}\, \mathcal{F} f^* \quad \forall\, f \in \mathrm{L}^1(\mathbb{R}^n),$$

where $^* : \mathrm{L}^1(\mathbb{R}^n) \longrightarrow \mathrm{L}^1(\mathbb{R}^n)$ denotes the parity operator, defined as

$$f^*(x) = f(-x) \quad \text{for } x \in \mathbb{R}^n.$$

Since $\mathcal{F}w \in \mathrm{L}^1(\mathbb{R}^2)$, the Fourier slice theorem gives

$$\begin{aligned} \mathcal{F}^{-1}\big(\mathcal{R}(\mathcal{F}w)\big)(t, \theta) &= (2\pi)^{-1}\, \mathcal{F}\big((\mathcal{R}(\mathcal{F}w))^*\big)(t, \theta) = (2\pi)^{-1}\, \mathcal{F}\big(\mathcal{R}((\mathcal{F}w)^*)\big)(t, \theta) \\ &= (2\pi)^{-1}\, \mathcal{F}\big(\mathcal{R}((2\pi)^2\, \mathcal{F}^{-1}w)\big)(t, \theta) = 2\pi\, \mathcal{F}\big(\mathcal{R}(\mathcal{F}^{-1}w)\big)(t, \theta) \\ &= 2\pi\, \mathcal{F}(\mathcal{F}^{-1}w)(t\cos(\theta), t\sin(\theta)) = 2\pi\, w(t\cos(\theta), t\sin(\theta)) \end{aligned}$$

for any $(t, \theta) \in \mathbb{R} \times [0, \pi)$, by using the Fourier inversion formula on $\mathcal{S}(\mathbb{R}^2)$. So we finally obtain

$$\begin{aligned} \langle R_L g, \mathcal{F}w \rangle &= \frac{1}{2\pi} \int_0^\pi \int_{\mathbb{R}} A_L(t)\, \mathcal{F}g(t, \theta)\, 2\pi\, w(t\cos(\theta), t\sin(\theta))\, \mathrm{d}t\, \mathrm{d}\theta \\ &= \int_0^\pi \int_{\mathbb{R}} W_L(t)\, \mathcal{F}g(t, \theta)\, w(t\cos(\theta), t\sin(\theta))\, |t|\, \mathrm{d}t\, \mathrm{d}\theta. \end{aligned}$$

Transforming back to Cartesian coordinates, i.e., $(x, y) = (t\cos(\theta), t\sin(\theta))$, we have

$$\mathcal{F}(R_L g)(S\cos(\theta), S\sin(\theta)) = W_L(S)\, \mathcal{F}g(S, \theta) \quad \text{for almost all } (S, \theta) \in \mathbb{R} \times [0, \pi).$$

Since $W \in \mathrm{L}^\infty(\mathbb{R})$ is compactly supported with $\mathrm{supp}(W) \subseteq [-1, 1]$ and $g \in \mathrm{L}^2(\mathbb{R} \times [0, \pi))$, we can conclude that $\mathcal{F}(R_L g) \in \mathrm{L}^2(\mathbb{R}^2)$. Indeed, from transformation to polar coordinates we obtain

$$\begin{aligned} \|\mathcal{F}(R_L g)\|^2_{\mathrm{L}^2(\mathbb{R}^2)} &= \int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{F}(R_L g)(X, Y)|^2\, \mathrm{d}X\, \mathrm{d}Y \\ &= \int_0^\pi \int_{\mathbb{R}} |\mathcal{F}(R_L g)(S\cos(\theta), S\sin(\theta))|^2\, |S|\, \mathrm{d}S\, \mathrm{d}\theta \\ &= \int_0^\pi \int_{\mathbb{R}} |W_L(S)|^2\, |S|\, |\mathcal{F}g(S, \theta)|^2\, \mathrm{d}S\, \mathrm{d}\theta. \end{aligned}$$

Because the scaled window function W_L has compact support in $[-L, L]$, we finally obtain

$$\|\mathcal{F}(R_L g)\|^2_{\mathrm{L}^2(\mathbb{R}^2)} \leq \left(\sup_{S \in [-L,L]} |S| \, |W_L(S)|^2 \right) \int_0^\pi \int_{\mathbb{R}} |\mathcal{F}g(S,\theta)|^2 \, \mathrm{d}S \, \mathrm{d}\theta$$
$$= 2\pi \, L \left(\sup_{S \in [-1,1]} |S| \, |W(S)|^2 \right) \|g\|^2_{\mathrm{L}^2(\mathbb{R} \times [0,\pi)} < \infty.$$

By the Rayleigh-Plancherel theorem, we also have $R_L g \in \mathrm{L}^2(\mathbb{R}^2)$ with

$$\|R_L g\|^2_{\mathrm{L}^2(\mathbb{R}^2)} = \frac{1}{4\pi^2} \, \|\mathcal{F}(R_L g)\|^2_{\mathrm{L}^2(\mathbb{R}^2)} \leq \frac{L}{2\pi} \left(\sup_{S \in [-1,1]} |S| \, |W(S)|^2 \right) \|g\|^2_{\mathrm{L}^2(\mathbb{R} \times [0,\pi)},$$

i.e.,

$$\|R_L g\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \frac{1}{\sqrt{2\pi}} \left(\sup_{S \in [-1,1]} |S| \, |W(S)|^2 \right)^{1/2} L^{1/2} \, \|g\|_{\mathrm{L}^2(\mathbb{R} \times [0,\pi))},$$

which completes our proof. □

We are now in a position, where we can analyse the data error $f_L - f_L^\delta$ in the L^2-norm for target functions $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ with some $\alpha > 0$ satisfying $\mathcal{R}f \in \mathrm{L}^2(\mathbb{R} \times [0,\pi))$, where

$$f_L = \frac{1}{2} \, \mathcal{B}\big(q_L * \mathcal{R}f\big) \quad \text{and} \quad f_L^\delta = \frac{1}{2} \, \mathcal{B}\big(q_L * g^\delta\big)$$

with noisy measurements $g^\delta \in \mathrm{L}^1(\mathbb{R} \times [0,\pi)) \cap \mathrm{L}^2(\mathbb{R} \times [0,\pi))$.

Theorem 8.6. *For $\alpha > 0$ let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ satisfy $\mathcal{R}f \in \mathrm{L}^2(\mathbb{R} \times [0,\pi))$. Further, for $\delta > 0$ let $g^\delta \in \mathrm{L}^1(\mathbb{R} \times [0,\pi)) \cap \mathrm{L}^2(\mathbb{R} \times [0,\pi))$ satisfy*

$$\|\mathcal{R}f - g^\delta\|_{\mathrm{L}^2(\mathbb{R} \times [0,\pi))} \leq \delta.$$

Then, the L^2-norm of the data error $f_L - f_L^\delta$ is bounded above by

$$\|f_L - f_L^\delta\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq c_W \, L^{1/2} \, \delta,$$

where

$$c_W^2 = \frac{1}{2\pi} \sup_{S \in [-1,1]} |S| \, |W(S)|^2.$$

Proof. Due to Lemma 8.1, $f \in \mathrm{L}^1(\mathbb{R}^2)$ implies $\mathcal{R}f \in \mathrm{L}^1(\mathbb{R} \times [0,\pi))$. Moreover, $\mathcal{R}f \in \mathrm{L}^2(\mathbb{R} \times [0,\pi))$ and $g^\delta \in \mathrm{L}^1(\mathbb{R} \times [0,\pi)) \cap \mathrm{L}^2(\mathbb{R} \times [0,\pi))$ by assumption. This allows us to use the linear regularization operator $R_L : \mathrm{L}^1(\mathbb{R} \times [0,\pi)) \cap \mathrm{L}^2(\mathbb{R} \times [0,\pi)) \longrightarrow \mathrm{L}^2(\mathbb{R}^2)$, satisfying

$$R_L g = \frac{1}{2} \, \mathcal{B}\big(q_L * g\big),$$

to obtain the representation

$$f_L - f_L^\delta = R_L(\mathcal{R}f) - R_L g^\delta = R_L(\mathcal{R}f - g^\delta) \in \mathrm{L}^2(\mathbb{R}^2).$$

Finally, by using Theorem 8.5, this gives the error estimate

$$\|f_L - f_L^\delta\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \frac{1}{\sqrt{2\pi}} \left(\sup_{S \in [-1,1]} |S| \, |W(S)|^2 \right)^{1/2} L^{1/2} \, \|\mathcal{R}f - g^\delta\|_{\mathrm{L}^2(\mathbb{R} \times [0,\pi))} \leq c_W \, L^{1/2} \, \delta,$$

as stated. □

8.2. **Analysis of the approximation error.** For convenience, we recall two relevant estimates on the approximation error $e_L = f - f_L$ from §5 and §7, which we use in the analysis of the overall FBP reconstruction error. We first rely on the basic assumption that the smallest maximizer $S^*_{\alpha,W,L} \in [0,1]$ of the even function

$$\Phi_{\alpha,W,L}(S) = \frac{(1-W(S))^2}{(1+L^2S^2)^\alpha} \quad \text{for } S \in [-1,1].$$

is uniformly bounded away from 0, i.e., there exists a constant $c_{\alpha,W} > 0$ satisfying

$$\text{(A)} \qquad S^*_{\alpha,W,L} \geq c_{\alpha,W} \quad \forall\, L > 0.$$

Theorem 8.7 (see Theorem 5.5)**.** *For $\alpha > 0$ let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ and let $W \in \mathcal{C}([-1,1])$ satisfy $W(0) = 1$. Then, the L^2-norm of the approximation error $f - f_L$ is under the assumption (A) bounded above by*

$$\|f - f_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \big(c_{\alpha,W}^{-\alpha}\, \|1-W\|_{\infty,[-1,1]} + 1\big) L^{-\alpha}\, \|f\|_\alpha.$$

Our second L^2-error estimate on e_L from §7 works only with conditions on W, stated as follows.

Theorem 8.8 (see Corollary 7.2)**.** *For $\alpha > 0$ let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$ and let $W \in \mathcal{C}^k([-1,1])$, for $k \geq 2$, satisfy*

$$W(0) = 1 \quad \text{and} \quad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1.$$

Then, the L^2-norm of the approximation error $f - f_L$ is bounded above by

$$\|f - f_L\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \big(c_{\alpha,k}\, \|W^{(k)}\|_{\infty,[-1,1]} + 1\big)\, L^{-\min\{k,\alpha\}}\, \|f\|_\alpha$$

with some constant $c_{\alpha,k} > 0$ independent of W and f.

8.3. **Analysis of the overall FBP reconstruction error.** Starting from the decomposition

$$\|e_L^\delta\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \|f - f_L\|_{\mathrm{L}^2(\mathbb{R}^2)} + \|f_L - f_L^\delta\|_{\mathrm{L}^2(\mathbb{R}^2)}$$

we combine the results of this section to estimate the FBP reconstruction error e_L^δ in (8.2).

On the one hand, combining Theorem 8.6 with Theorem 8.7, gives the estimate

$$\|e_L^\delta\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \big(c_{\alpha,W}^{-\alpha}\, \|1-W\|_{\infty,[-1,1]} + 1\big) L^{-\alpha}\, \|f\|_\alpha + c_W\, L^{1/2}\, \delta.$$

By coupling the bandwidth L with the noise level δ via $L = \delta^{-\frac{2}{2\alpha+1}}\, \|f\|_\alpha^{\frac{2}{2\alpha+1}}$ we obtain

$$\|e_L^\delta\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \big(c_{\alpha,W}^{-\alpha}\, \|1-W\|_{\infty,[-1,1]} + c_W + 1\big)\, \|f\|_\alpha^{\frac{1}{2\alpha+1}}\, \delta^{\frac{2\alpha}{2\alpha+1}},$$

i.e.,

$$\|f - f_L^\delta\|_{\mathrm{L}^2(\mathbb{R}^2)} = \mathcal{O}\Big(\delta^{\frac{2\alpha}{2\alpha+1}}\Big) \quad \text{for} \quad \delta \searrow 0.$$

This gives our first result concerning convergence rates for noisy data.

Corollary 8.9 (Convergence rates for noisy data I)**.** *Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for $\alpha > 0$, satisfy $\mathcal{R}f \in \mathrm{L}^2(\mathbb{R} \times [0,\pi))$. Furthermore, suppose $W \in \mathcal{C}([-1,1])$ with $W(0) = 1$ and, moreover, let $g^\delta \in \mathrm{L}^1(\mathbb{R} \times [0,\pi)) \cap \mathrm{L}^2(\mathbb{R} \times [0,\pi))$ satisfy $\|\mathcal{R}f - g^\delta\|_{\mathrm{L}^2(\mathbb{R}\times[0,\pi))} \leq \delta$. Then, the L^2-norm of the FBP reconstruction error $e_L^\delta = f - f_L^\delta$ is under assumption* (A) *bounded above by*

$$\|e_L^\delta\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \big(c_{\alpha,W}^{-\alpha}\, \|1-W\|_{\infty,[-1,1]} + c_W + 1\big)\, \|f\|_\alpha^{\frac{1}{2\alpha+1}}\, \delta^{\frac{2\alpha}{2\alpha+1}},$$

where $L = \delta^{-\frac{2}{2\alpha+1}}\, \|f\|_\alpha^{\frac{2}{2\alpha+1}}$. In particular, we have

$$\|e_L^\delta\|_{\mathrm{L}^2(\mathbb{R}^2)} = \mathcal{O}\Big(\delta^{\frac{2\alpha}{2\alpha+1}}\Big) \quad \textit{for} \quad \delta \searrow 0.$$

On the other hand, the combination of Theorem 8.6 and Theorem 8.8 yields the estimate

$$\|e_L^\delta\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \big(c_{\alpha,k}\,\|W^{(k)}\|_{\infty,[-1,1]} + 1\big)\, L^{-\min\{k,\alpha\}}\,\|f\|_\alpha + c_W\, L^{1/2}\,\delta.$$

By choosing $L = \delta^{-\frac{2}{2\min\{k,\alpha\}+1}}\,\|f\|_\alpha^{\frac{2}{2\min\{k,\alpha\}+1}}$ we now obtain

$$\|e_L^\delta\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \big(c_{\alpha,k}\,\|W^{(k)}\|_{\infty,[-1,1]} + c_W + 1\big)\,\|f\|_\alpha^{\frac{1}{2\min\{k,\alpha\}+1}}\,\delta^{\frac{2\min\{k,\alpha\}}{2\min\{k,\alpha\}+1}},$$

i.e.,

$$\|f - f_L^\delta\|_{\mathrm{L}^2(\mathbb{R}^2)} = \mathcal{O}\Big(\delta^{\frac{2\min\{k,\alpha\}}{2\min\{k,\alpha\}+1}}\Big) \quad \text{for} \quad \delta \searrow 0.$$

This finally yields another result concerning convergence rates for noisy data.

Corollary 8.10 (Convergence rates for noisy data II)**.** *Let $f \in \mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{H}^\alpha(\mathbb{R}^2)$, for $\alpha > 0$, satisfy $\mathcal{R}f \in \mathrm{L}^2(\mathbb{R} \times [0,\pi))$. Moreover, suppose that $W \in \mathcal{C}^k([-1,1])$, for $k \geq 2$, with*

$$W(0) = 1 \quad \text{and} \quad W^{(j)}(0) = 0 \quad \forall\, 1 \leq j \leq k-1,$$

and let $g^\delta \in \mathrm{L}^1(\mathbb{R} \times [0,\pi)) \cap \mathrm{L}^2(\mathbb{R} \times [0,\pi))$ satisfy $\|\mathcal{R}f - g^\delta\|_{\mathrm{L}^2(\mathbb{R}\times[0,\pi))} \leq \delta$. Then, the L^2-norm of the FBP reconstruction error $e_L^\delta = f - f_L^\delta$ is bounded above by

$$\|e_L^\delta\|_{\mathrm{L}^2(\mathbb{R}^2)} \leq \big(c_{\alpha,k}\,\|W^{(k)}\|_{\infty,[-1,1]} + c_W + 1\big)\,\|f\|_\alpha^{\frac{1}{2\min\{k,\alpha\}+1}}\,\delta^{\frac{2\min\{k,\alpha\}}{2\min\{k,\alpha\}+1}},$$

where $L = \delta^{-\frac{2}{2\min\{k,\alpha\}+1}}\,\|f\|_\alpha^{\frac{2}{2\min\{k,\alpha\}+1}}$. In particular, we have

$$\|e_L^\delta\|_{\mathrm{L}^2(\mathbb{R}^2)} = \mathcal{O}\Big(\delta^{\frac{2\min\{k,\alpha\}}{2\min\{k,\alpha\}+1}}\Big) \quad \textit{for} \quad \delta \searrow 0.$$

9. Conclusion

We have analysed the inherent FBP reconstruction error which is incurred by the use of a low-pass filter with a compactly supported window W and finite bandwidth L. We refined our L^2-error estimate from [1] to prove, under reasonable assumptions, convergence of the FBP reconstruction f_L to the target function f as the bandwidth L goes to infinity. Moreover, we developed asymptotic convergence rates in terms of the bandwidth L and the smoothness α of the target function f.

By deriving an asymptotic error estimate, we observed that the flatness of the filter's window function is of fundamental importance. Indeed, if the window W is k-times differentiable at the origin, such that the first $k-1$ derivatives vanish at zero, then the convergence rate of the obtained error bound saturates at $\mathcal{O}(L^{-k})$, and the quantity $|W^{(k)}(0)|$ determines the approximation quality of the chosen low-pass filter. The estimates provided for the approximation error can be combined with error estimates on the data error to obtain convergence rates for noisy data.

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